1

Lecture Notes

An Introduction to Riemannian Geometry

(version 1.102 - September 1996)

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Preface

These lecture notes grew out of an M.Sc.-course on differential geometry which I gave at the University of Leeds 1992. Their main purpose is to introduce the beautiful theory of Riemannian Geometry a still very active area of Mathematics. This is a subject with no lack of interesting examples. They are indeed the key to a good understanding of it and will therefore play a major role throughout this work. Of special interest are the classical Lie groups allowing concrete calculations of many of the abstract notions on the menu.

The study of Riemannian Geometry is rather meaningless without some basic knowledge on Gaussian Geometry i.e. the differential geometry of curves and surfaces in Euclidean 3-space. For this we recommend the excellent textbook: M. P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall (1976).

These lecture notes are written for students with a good understanding of linear algebra, real analysis of several variables, the classical theory of ODEs and some topology. The most important results stated in the text are also proved there. Other smaller ones are left to the reader as exercises, which follow at the end of each chapter. This format is aimed at students willing to put **hard work** into the course.

It is my intention to extent this very incomplete first draft, which unfortunately still contains typing errors, and include some of the differential geometry of the Riemannian symmetric spaces.

For further reading we recommend the very interesting textbook: M. P. do Carmo, *Riemannian Geometry*, Birkhäuser (1992).

Lund University, May 1996

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Contents

CHAPTER 1

Introduction

On the 10th of June 1854 Georg Friedrich Bernhard Riemann gave his famous "Habilitationsvortrag" in the Colloquium of the Philosophical Faculty at Göttingen. His talk with the title "Über die Hypothesen, welche der Geometrie zu Grunde liegen" is often said to be the most important in the history of differential geometry. Gauss, at the age of 76, was in the audience and is said to have been very impressed.

Riemann's revolutionary ideas generalized the geometry of surfaces which had been studied earlier by Gauss, Bolyai and Lobachevsky. Later they lead to an exact definition of the modern concept of an abstract n-dimensional Riemannian manifold.

CHAPTER 2

Differentiable Manifolds

The main purpose of this chapter is to introduce the concept of a differentiable manifold, generalizing the idea of a differentiable surface studied in most introductory courses on Differential Geometry. Furthermore we study submanifolds and differentiable maps between manifolds.

Definition 2.1. Let M be a topological Hausdorff space with a countable basis. M is called a **topological manifold** if there exists an $m \in \mathbb{N}$ and for every point $p \in M$ an open neighbourhood U_p of p, such that U_p is homeomorphic to some open subset V_p of $\mathbb{R}^m$. The natural number m is called the **dimension** of M. To denote that the dimension of M is m we write M^m.

For an open subset U of $\mathbb{R}^m$ and $r \in \mathbb{N}$ we denote by $C^r(U, \mathbb{R}^n)$ the r-times continuously differentiable maps from U to $\mathbb{R}^n$. By **smooth** we mean $C^\infty = \cap_{r=1}^\infty C^r$ and C^ω means **real analytic**.

Definition 2.2. Let M^m be a topological manifold, U be an open and connected subset of M and $\phi : U \to \mathbb{R}^m$ be a continuous map which is a homeomorphism onto its image $\phi(U)$. Then (U, ϕ) is called a **chart** (or **local coordinate**) on M. A collection

$$\mathcal{A} = \{(U_\alpha, \phi_\alpha)|\ \alpha \in I\}$$

of charts on M is called a C^r**-atlas** if

i. $M = \cup_\alpha U_\alpha$,

ii. The corresponding **transition maps**

$$\phi_\beta \circ \phi_\alpha^{-1}|_{\phi_\alpha(U_\alpha \cap U_\beta)} : \phi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{R}^m$$

are C^r for all $\alpha, \beta \in I$.

Definition 2.3. Let $\mathcal{A}$ be a C^r-atlas on M. A chart (U, ϕ) on M is said to be **compatible** with $\mathcal{A}$ if $\mathcal{A} \cup \{(U, \phi)\}$ is a C^r-atlas on M. A C^r-atlas $\hat{\mathcal{A}}$ is said to be **maximal** if it contains all charts compatible with it. A maximal atlas $\hat{\mathcal{A}}$ on M is also called a C^r**-structure** on M. A C^r**-manifold** is a topological manifold M with a C^r-structure. A manifold is said to be **smooth** if it is C^∞ and **real analytic** if it is C^ω.

Remark 2.4. Note that a C^r-atlas $\mathcal{A}$ on M determines a unique C^r-structure $\hat{\mathcal{A}}$ containing $\mathcal{A}$. $\hat{\mathcal{A}}$ consists of all charts compatible with $\mathcal{A}$.

Example 2.5. Let $\mathbb{R}^m$ be the real m-dimensional vector space with the usual topology $\mathcal{T}$ induced by the distance function

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_m - y_m)^2}.$$

For the topological space $(\mathbb{R}^m, \mathcal{T})$ we have the **trivial** C^ω-atlas

$$\mathcal{A} = \{(\mathbb{R}^m, \mathrm{id}_{\mathbb{R}^m}) |\ \mathrm{id}_{\mathbb{R}^m} : x \mapsto x\}$$

inducing the **standard** C^ω-structure $\hat{\mathcal{A}}$ on $\mathbb{R}^m$.

Example 2.6. Let S^m denote the unit sphere in $\mathbb{R}^{m+1}$ i.e.

$$S^m = \{x \in \mathbb{R}^{m+1} |\ x_1^2 + \ldots + x_{m+1}^2 = 1\}$$

equipped with the subset topology $\mathcal{T}_{S^m}$ induced by $\mathcal{T}$ on $\mathbb{R}^{m+1}$. Let n be the north pole $n = (1,0) \in \mathbb{R} \times \mathbb{R}^m$ and s be the south pole $s = (-1,0)$ on S^m, respectively. Put $U_n = S^m - \{n\}$, $U_s = S^m - \{s\}$ and define $\phi_n : U_n \to \mathbb{R}^m$, $\phi_s : U_s \to \mathbb{R}^m$ by

$$\phi_n : (x_1, \ldots, x_{m+1}) \mapsto \frac{1}{1 - x_1}(x_2, \ldots, x_{m+1}),$$

$$\phi_s : (x_1, \ldots, x_{m+1}) \mapsto \frac{1}{1 + x_1}(x_2, \ldots, x_{m+1}).$$

Then the transition maps $\phi_s \circ \phi_n^{-1}, \phi_n \circ \phi_s^{-1} : \mathbb{R}^m - \{0\} \to \mathbb{R}^m - \{0\}$ are given by $x \mapsto x/|x|^2$ so $\mathcal{A} = \{(U_n, \phi_n), (U_s, \phi_s)\}$ is a C^ω-atlas on S^m. The C^ω-manifold $(S^m, \hat{\mathcal{A}})$ is called the m-dimensional **standard sphere**.

Example 2.7. On the set $\mathbb{R}^{m+1} - \{0\}$ we define the equivalence relation $\equiv$ by

$$x \equiv y \text{ if and only if there exists } \lambda \in \mathbb{R}^* \text{ such that } \lambda x = y.$$

Let $\pi : \mathbb{R}^{m+1} - \{0\} \to (\mathbb{R}^{m+1} - \{0\})/\equiv$ be the natural projection $\pi : x \mapsto [x]$ onto the quotient space which we denote by $\mathbb{R}P^m$ and equip with the quotient topology induced by π and $\mathcal{T}$ on $\mathbb{R}^{m+1}$. For $k \in \{1, \ldots, m+1\}$ put $U_k = \{[x] \in \mathbb{R}P^m |\ x_k \neq 0\}$ and define $\phi_k : U_k \to \mathbb{R}^m$ by

$$\phi_k : [x] \mapsto (\frac{x_1}{x_k}, \ldots, \frac{x_{k-1}}{x_k}, \frac{x_{k+1}}{x_k}, \ldots, \frac{x_{m+1}}{x_k}).$$

If $[x] \equiv [y]$ then $\lambda x = y$ for some $\lambda \in \mathbb{R}^*$ so $x_l/x_k = y_l/y_k$ for all l. This means that ϕ_k is well defined for all k. The transition maps $\phi_k \circ \phi_l^{-1}|_{\phi_l(U_l \cap U_k)} : \phi_l(U_l \cap U_k) \to \mathbb{R}^m$ are given by

$$(\frac{x_1}{x_l}, \dots, \frac{x_{l-1}}{x_l}, \frac{x_{l+1}}{x_l}, \dots, \frac{x_{m+1}}{x_l}) \mapsto (\frac{x_1}{x_k}, \dots, \frac{x_{k-1}}{x_k}, \frac{x_{k+1}}{x_k}, \dots, \frac{x_{m+1}}{x_k})$$

so $\mathcal{A} = \{(U_k, \phi_k)|\ k = 1, \dots, m+1\}$ is a C^ω-atlas on $\mathbb{R}P^m$. The manifold $(\mathbb{R}P^m, \hat{\mathcal{A}})$ is called the m-dimensional **real projective space**.

Example 2.8. Put $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\mathbb{C}^* = \mathbb{C} - \{0\}$, $U_0 = \mathbb{C}$ and $U_\infty = \hat{\mathbb{C}} - \{0\}$. Then define the charts $\phi_0 : U_0 \to \mathbb{C}$, $\phi_\infty : U_\infty \to \mathbb{C}$ by $\phi_0 : z \mapsto z$ and $\phi_\infty : w \mapsto 1/w$, respectively. Then the transition maps $\phi_\infty \circ \phi_0^{-1}, \phi_0 \circ \phi_\infty^{-1} : \mathbb{C}^* \to \mathbb{C}^*$ are given by $z \mapsto 1/z$ so $\mathcal{A} = \{(U_0, \phi_0), (U_\infty, \phi_\infty)\}$ is a C^ω-atlas on $\hat{\mathbb{C}}$. The C^ω-manifold $(\hat{\mathbb{C}}, \hat{\mathcal{A}})$ is called the **Riemann sphere**.

In what follows we shall by $\mathbb{R}^m$, S^m, $\mathbb{R}P^m$ or $\hat{\mathbb{C}}$ always mean the standard manifolds above.

We now define the concept of a submanifold $\tilde{M}$ of $(M^m, \hat{\mathcal{A}})$. Submanifolds are very important objects which will be given considerable attention as we go along. The additional structures that we will introduce on $(M^m, \hat{\mathcal{A}})$ give the corresponding induced structures on $\tilde{M}$ in a natural manner. We shall often be interested in how they are related.

Definition 2.9. Let $(M^m, \hat{\mathcal{A}})$ be a C^r-manifold and $\tilde{M}$ be a subset of M equipped with the subset topology of M. The set $\tilde{M}$ is said to be an n-dimensional **submanifold** of M if for each $p \in \tilde{M}$ there exists a chart $(U_p, \phi_p) \in \hat{\mathcal{A}}$ such that

i. $p \in U_p$
ii. $\phi_p : U_p \to \mathbb{R}^n \times \mathbb{R}^{m-n}$ satisfies

$$\phi_p(U_p \cap \tilde{M}) = \phi_p(U_p) \cap (\mathbb{R}^n \times \{0\}).$$

The positive natural number $(m - n)$ is called the **codimension** of $\tilde{M}$ in M^m.

Proposition 2.10. *Let $(M^m, \hat{\mathcal{A}})$ be a C^r-manifold and $\tilde{M}$ be an n-dimensional submanifold of $(M^m, \hat{\mathcal{A}})$. For every point $p \in \tilde{M}$ let $(U_p, \phi_p) \in \hat{\mathcal{A}}$ be a chart on M such that $p \in U_p$ and the map $\phi_p : U_p \to \mathbb{R}^n \times \mathbb{R}^{m-n}$ satisfies*

$$\phi_p(U_p \cap \tilde{M}) = \phi_p(U_p) \cap (\mathbb{R}^n \times \{0\}).$$

Further let $\pi : \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^n$ be the natural projection onto the first factor. Then

$$\mathcal{B} = \{(U_p \cap \tilde{M}, \pi \circ \phi_p|_{U_p \cap \tilde{M}})|\ p \in \tilde{M}\}$$

is a C^r-atlas for $\tilde{M}$. In particular $(\tilde{M}, \hat{\mathcal{B}})$ is an n-dimensional C^r-manifold.

Proof. See Exercise 2.2. □

We shall now see how the well-known Inverse Function Theorem for finite dimensional real vector spaces can be used to construct differentiable manifolds as submanifolds of $\mathbb{R}^m$.

Fact 2.11 (The Inverse Function Theorem). *Let U be an open subset of $\mathbb{R}^m$ and $F : U \to \mathbb{R}^m$ be a C^r-map. If $p \in U$ and the derivative $DF_p : \mathbb{R}^m \to \mathbb{R}^m$ of F at p is invertible, then there exist open neighbourhoods V around p and W around $F(p)$ such that $\hat{F} = F|_V : V \to W$ is bijective and the inverse $(\hat{F})^{-1} : W \to V$ is a C^r-map. The derivative $D(\hat{F}^{-1})_{F(p)}$ of $\hat{F}^{-1}$ at $F(p)$ is given by*

$$D(\hat{F}^{-1})_{F(p)} = (DF_p)^{-1}.$$

Definition 2.12. Let U be an open subset of $\mathbb{R}^m$ and $F : U \to \mathbb{R}^n$ be a C^r-map. A point $p \in U$ is called a **critical point** for F if $DF_p : \mathbb{R}^m \to \mathbb{R}^n$ is not of full rank, and a **regular point** if it is not critical. A point $q \in F(U)$ is called a **regular value** of F if every point $p \in F^{-1}(\{q\})$ is a regular point for F and a **critical value** otherwise.

Theorem 2.13 (The Implicit Function Theorem). *Let $m > n$ and $F : U \to \mathbb{R}^n$ be a C^r-map from an open subset U of $\mathbb{R}^m$. If $q \in F(U)$ is a regular value of F then $F^{-1}(\{q\})$ is an $(m-n)$-dimensional submanifold of $\mathbb{R}^m$.*

Proof. Let p be an element of $F^{-1}(\{q\})$ and K_p be the kernel of the derivative DF_p i.e. the $(m-n)$-dimensional subspace of $\mathbb{R}^m$ given by $K_p = \{v \in \mathbb{R}^m |\ DF_p \cdot v = 0\}$. Let $\pi_p : \mathbb{R}^m \to \mathbb{R}^{m-n}$ be a linear map such that $\pi_p|_{K_p} : K_p \to \mathbb{R}^{m-n}$ is bijective and define $G_p : U \to \mathbb{R}^n \times \mathbb{R}^{m-n}$ by $G_p : x \mapsto (F(x), \pi_p(x))$.

The derivative $(DG_p)_p : \mathbb{R}^m \to \mathbb{R}^m$ of G_p is, with respect to the decomposition $\mathbb{R}^m = K_p^\perp \oplus K_p$, given by

$$DG_p = \begin{pmatrix} DF_p \\ (0, \pi_p) \end{pmatrix}$$

so it is bijective. It now follows from the inverse function theorem that there exist open neighbourhoods V_p around p and W_p around $G_p(p)$ such that $\hat{G}_p = G_p|_{V_p} : V_p \to W_p$ is bijective. The inverse $\hat{G}_p^{-1} : W_p \to V_p$ is C^r and $D(\hat{G}_p^{-1})_{G_p(p)} = (DG_p)_p^{-1}$ so that $D(\hat{G}_p^{-1})_y$ is bijective for all $y \in W_p$. Now put $\tilde{U}_p = F^{-1}(\{q\}) \cap V_p$ then

$$\tilde{U}_p = \hat{G}_p^{-1}((\{q\} \times \mathbb{R}^{m-n}) \cap W_p)$$

so if $\pi : \mathbb{R}^n \times \mathbb{R}^{m-n} \to \mathbb{R}^{m-n}$ is the natural projection onto the second factor, then the map

$$\tilde{\phi}_p = \pi \circ G_p : \tilde{U}_p \to (\{q\} \times \mathbb{R}^{m-n}) \cap W_p \to \mathbb{R}^{m-n}$$

is a local chart on the open neighbourhood $\tilde{U}_p$ of p. The point $q \in F(U)$ is a regular value so the set

$$\mathcal{B} = \{(\tilde{U}_p, \tilde{\phi}_p) | \; p \in F^{-1}(\{q\})\}$$

is a C^r-atlas for $F^{-1}(\{q\})$. □

Example 2.14. Let $F : \mathbb{R}^{m+1} \to \mathbb{R}$ be the C^ω-map given by

$$F : (x_1, ..., x_{m+1}) \mapsto \frac{1}{2} \sum_{i=1}^{m+1} x_i^2.$$

The derivative DF_x of F at x is given by $DF_x = [x_1, ..., x_{m+1}]$ so $(DF_x) \cdot (DF_x)^t = |x|^2 \in \mathbb{R}$. This means that $1/2 \in \mathbb{R}$ is a regular value of F so the fibre

$$S^m = \{x \in \mathbb{R}^{m+1} | \; |x|^2 = 1\} = F^{-1}(\{1/2\})$$

of F is an m-dimensional submanifold of $\mathbb{R}^{m+1}$. It is called the m-dimensional sphere.

Example 2.15. Let $F : \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to \mathbb{R}^2$ be the C^ω-map defined by $F : (p, v) \mapsto ((|p|^2 - 1)/2, \langle p, v \rangle)$. The derivative $DF_{(p,v)}$ of F at (p, v) is given by

$$DF_{(p,v)} = \begin{pmatrix} p & 0 \\ v & p \end{pmatrix}.$$

Hence $\det[DF \cdot (DF)^t] = |p|^2(|p|^2 + |v|^2) = (1 + |v|^2) > 0$ on $F^{-1}(\{0\})$. This implies that

$$F^{-1}(\{0\}) = \{(p, v) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} | \;\; |p|^2 = 1 \;\text{ and }\; \langle p, v \rangle = 0\}$$

which we denote by TS^m is a $2m$-dimensional submanifold of $\mathbb{R}^{2m+2}$. We shall see later that TS^m is what is called the tangent bundle of the m-dimensional sphere.

We are now interested in differentiable maps between two differentiable manifolds i.e. those which respect the differentiable structures of the manifolds involved.

Definition 2.16. Let $(M^m, \hat{\mathcal{A}})$ and $(N^n, \hat{\mathcal{B}})$ be two C^r-manifolds. A map $\psi : M^m \to N^n$ is said to be a **C^r-map** if for all charts $(U, \phi_1) \in \hat{\mathcal{A}}$ and $(V, \phi_2) \in \hat{\mathcal{B}}$ the maps

$$\phi_2 \circ \psi \circ \phi_1^{-1}|_{\phi_1(U \cap \psi^{-1}(V))} : \phi_1(U \cap \psi^{-1}(V)) \to \mathbb{R}^n$$

are of class C^r. A C^r-map $f : (M^m, \hat{\mathcal{A}}) \to \mathbb{R}$ is called a C^r-**function** on M.

Proposition 2.17. *Let $\psi_1 : (M, \hat{\mathcal{A}}) \to (\hat{N}, \hat{\mathcal{C}})$ and $\psi_2 : (\hat{N}, \hat{\mathcal{C}}) \to (N, \hat{\mathcal{B}})$ be two C^r-maps, then the composition $\psi_2 \circ \psi_1 : (M, \hat{\mathcal{A}}) \to (N, \hat{\mathcal{B}})$ is also a C^r-map.*

PROOF. See exercise 2.7. □

Example 2.18. It is easily seen that the following maps are C^ω i.e. real analytic, see Exercise 2.8.

i. $\phi_1 : S^2 \subset \mathbb{R}^3 \to S^3 \subset \mathbb{R}^4$, $\phi_1 : (x, y, z) \mapsto (x, y, z, 0)$.
ii. $\phi_2 : S^3 \subset \mathbb{C}^2 \to S^2 \subset \mathbb{C} \times \mathbb{R}$, $\phi_2 : (z_1, z_2) \mapsto (2z_1\bar{z}_2, |z_1|^2 - |z_2|^2)$.
iii. $\phi_3 : \mathbb{R}^1 \to S^1 \subset \mathbb{C}$, $\phi_3 : t \mapsto e^{it}$.
iv. $\phi_4 : S^m \to \mathbb{R}P^m$, $\phi_4 : x \mapsto [x]$.
v. $\phi_5 : \mathbb{R}^{m+1} - \{0\} \to \mathbb{R}P^m$, $\phi_5 : x \mapsto [x]$.
vi. $\phi_6 : \mathbb{R}^{m+1} - \{0\} \to S^m$, $\phi_6 : x \mapsto x/|x|$.

Example 2.19. Let $\mathbb{R}^{m\times m}$ denote the set of all real $m\times m$-matrices and $\text{Sym}(\mathbb{R}^m)$ be the subset of those which are symmetric i.e.

$$\text{Sym}(\mathbb{R}^m) = \{A \in \mathbb{R}^{m\times m} |\ A = A^t\}.$$

Then $\text{Sym}(\mathbb{R}^m)$ can be identified with $\mathbb{R}^n$ where $n = m(m+1)/2$. Define $F : \mathbb{R}^{m\times m} \to \text{Sym}(\mathbb{R}^m)$ by $F : A \mapsto AA^t$. The differential DF_A of F at $A \in \mathbb{R}^{m\times m}$ is given by $DF_A(X) = AX^t + XA^t \in \text{Sym}(\mathbb{R}^m)$. For $A \in \mathbf{O}(m) = F^{-1}(\{e\}) = \{A \in \mathbb{R}^{m\times m} |\ AA^t = e\}$ and $Y \in \text{Sym}(\mathbb{R}^m)$ we see that $DF_A(YA) = 2Y$ so DF_A is surjective for every $A \in \mathbf{O}(m)$. Hence e is a regular value of F and following the implicit function theorem $\mathbf{O}(m)$ is an $m(m-1)/2$-dimensional C^ω-submanifold of $\mathbb{R}^{m\times m} \cong \mathbb{R}^{m^2}$. The set $\mathbf{O}(m)$ is the well known orthogonal group and the usual matrix multiplication $\cdot$ is a group structure on it. It is easily checked that the map $\rho : \mathbf{O}(m)\times\mathbf{O}(m) \to \mathbf{O}(m)$ with $\rho : (x, y) \mapsto x\cdot y^{-1}$ is C^ω.

Definition 2.20. A **Lie group** is a C^ω-manifold G with a group structure $*$ such that the map $\rho : G \times G \to G$ with $\rho : (x, y) \mapsto x * y^{-1}$ is C^ω.

Example 2.21. Let $+$ denote the usual addition in $\mathbb{R}^m$. Then the pair $(\mathbb{R}^m, +)$ is an abelian Lie group.

Example 2.22. Let $\cdot$ denote the usual multiplication of the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$ or the quaternions $\mathbb{H}$. Then $(\mathbb{R}^*, \cdot)$, $(\mathbb{C}^*, \cdot)$, and $(\mathbb{H}^*, \cdot)$ are Lie groups. The corresponding unit spheres S^0, S^1 and S^3 are compact Lie subgroups in the trivial way.

Example 2.23. We now give a few examples of matrix Lie groups. In all cases the operation $\cdot$ is the usual matrix multiplication.

i. $\mathbf{GL}(\mathbb{R}^m) = \{A \in \mathbb{R}^{m\times m} |\det A \neq 0\}$ is called the real general linear group,
ii. $\mathbf{SL}(\mathbb{R}^m) = \{A \in \mathbb{R}^{m\times m} |\det A = 1\}$ is called the real special linear group,
iii. $\mathbf{O}(m) = \{A \in \mathbb{R}^{m\times m} | A^t \cdot A = e\}$ is called the orthogonal group,
iv. $\mathbf{SO}(m) = \{A \in \mathbf{O}(m) | \det A = 1\}$ is called the special orthogonal group,
v. $\mathbf{GL}(\mathbb{C}^m) = \{A \in \mathbb{C}^{m\times m} |\det A \neq 0\}$ is called the complex general linear group,
vi. $\mathbf{SL}(\mathbb{C}^m) = \{A \in \mathbb{C}^{m\times m} |\det A = 1\}$ is called the complex special linear group,
vii. $\mathbf{U}(m) = \{A \in \mathbb{C}^{m\times m} | \bar{A}^t \cdot A = e\}$ is called the unitary group, and
viii. $\mathbf{SU}(m) = \{A \in \mathbf{U}(m) | \det A = 1\}$ is called the special unitary group.

Definition 2.24. Two C^r-manifolds $(M, \hat{\mathcal{A}})$ and $(N, \hat{\mathcal{B}})$ are said to be **diffeomorphic** if there exists a bijective C^r-map $\psi : (M, \hat{\mathcal{A}}) \to (N, \hat{\mathcal{B}})$, such that its inverse $\psi^{-1} : (N, \hat{\mathcal{B}}) \to (M, \hat{\mathcal{A}})$ also is C^r. The map ψ is called a **diffeomorphism** between $(M, \hat{\mathcal{A}})$ and $(N, \hat{\mathcal{B}})$.

Definition 2.25. Let $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ be two C^r-structures on the same topological manifold M. $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ are said to be **different** if the identity map $\mathrm{id}_M : (M, \hat{\mathcal{A}}) \to (M, \hat{\mathcal{B}})$ is not a diffeomorphism.

Deep Result 2.26. *Let $(M^m, \hat{\mathcal{A}})$, $(N^n, \hat{\mathcal{B}})$ be C^r-manifolds of the same dimension i.e. $m = n$. If M and N are homeomorphic as topological spaces and $m \leq 3$ then $(M, \hat{\mathcal{A}})$ and $(N, \hat{\mathcal{B}})$ are diffeomorphic.*

The following remarkable result was proved by J.Milnor in his famous paper: *Differentiable structures on spheres*, Amer. J. Math. **81** (1959), 962-972.

Deep Result 2.27. *The seven dimensional sphere S^7 has exactly 28 different differentiable structures.*

Exercises

Exercise 2.1. Let $(M_1, \hat{\mathcal{A}}_1)$ and $(M_2, \hat{\mathcal{A}}_2)$ be two smooth manifolds and $M = (M_1 \times M_2, \mathcal{T})$ be the product space with the product topology. Find a C^∞-structure on M.

Exercise 2.2. Find a proof for Proposition 2.10.

Exercise 2.3. Let S^1 be the 1-dimensional sphere given by $\{z \in \mathbb{C} |\ |z|^2 = 1\}$. Use the maps $\phi_1 : \mathbb{C} - \{-1\} \to \mathbb{C}$ with $\phi_1 : z \mapsto (z-1)/(z+1)$ and $\phi_2 : \mathbb{C} - \{1\} \to \mathbb{C}$ with $\phi_2 : z \mapsto (1+z)/(1-z)$ to show that S^1 is a 1-dimensional submanifold of $\mathbb{C} = \mathbb{R}^2$.

Exercise 2.4. Show that the m-dimensional **torus**

$$T^m = \{z \in \mathbb{C}^m |\ |z_1| = \dots = |z_m| = 1\}$$

is an analytic submanifold of $\mathbb{C}^m \cong \mathbb{R}^{2m}$.

Exercise 2.5. Let $F : \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to \mathbb{R}^2$ be the analytic map given by

$$F : (x, y) \mapsto \frac{1}{2}(|x|^2 + |y|^2, |x|^2 - |y|^2)$$

and $\mathcal{R}$ be the set of critical values of F. Determine whether the sets $F^{-1}(\{(1,0)\})$ and $\mathcal{R}$ are submanifolds of their ambient spaces.

Exercise 2.6. Let $(G, *)$ and $(H, \cdot)$ be two Lie groups. Prove that the product manifold $G \times H$ has a Lie group structure.

Exercise 2.7. Find a proof of Proposition 2.17.

Exercise 2.8. Let $(M, \hat{\mathcal{A}})$ and $(N, \hat{\mathcal{B}})$ be two smooth manifolds and $\tilde{M}$, $\tilde{N}$ be submanifolds of M and N, respectively. Prove that if $\phi : M \to N$ is a smooth map with $\phi(\tilde{M})$ contained in $\tilde{N}$, then the restriction $\tilde{\phi} = \phi|_{\tilde{M}} : \tilde{M} \to \tilde{N}$ is smooth.

Exercise 2.9. Define two C^ω-structures $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ on $(\mathbb{R}, \mathcal{T})$ by the following atlases $\mathcal{A} = \{(\mathbb{R}, \mathrm{id}_{\mathbb{R}}) |\ \mathrm{id}_{\mathbb{R}} : x \mapsto x\}$ and $\mathcal{B} = \{(\mathbb{R}, \psi) |\ \psi : x \mapsto x^3\}$.

i. Is the chart $(\mathbb{R}, \psi)$ compatible with $\mathcal{A}$?

ii. Are $(\mathbb{R}, \hat{\mathcal{A}})$ and $(\mathbb{R}, \hat{\mathcal{B}})$ diffeomorphic?

Exercise 2.10. Prove that $(S^2, \{\pi_s, \pi_n\})$ and $(\hat{\mathbb{C}}, \{\phi_o, \phi_\infty\})$ as defined above are diffeomorphic.

Exercise 2.11. Prove the following diffeomorphies

$$S^1 \cong \mathbf{SO}(2), \quad S^3 \cong \mathbf{SU}(2),$$

$$\mathbf{SO}(n) \times \mathbf{O}(1) \cong \mathbf{O}(n), \quad \mathbf{SU}(n) \times \mathbf{U}(1) \cong \mathbf{U}(n).$$

CHAPTER 3

The Tangent Space

In this chapter we develope the idea of a tangent space from the theory of surfaces in $\mathbb{R}^3$ to the more general situation of a differentiable manifold M^m. We see a tangent vector X at a point p as a first order linear differential operator on the set of locally defined functions on the manifold or rather that of functions germs at p. We then prove that the tangent space T_pM i.e. the set of all tangent vectors at p is a vector space isomorphic to $\mathbb{R}^m$.

From now on we shall assume, when not stating otherwise, that our manifolds and maps are smooth i.e. in the C^∞-category. Let M^m be a manifold. For a point $p \in M$ let $\hat{\varepsilon}(p)$ be the set of all smooth functions defined on an open neighbourhood of p i.e.

$$\hat{\varepsilon}(p) = \{f : U_f \to \mathbb{R} |\ U_f \text{ is an open subset of } M \text{ containing } p\}.$$

On $\hat{\varepsilon}(p)$ we define the equivalence relation $\equiv$ by: $f \equiv g$ if and only if there exists an open neighbourhood $V \subset U_f \cap U_g$ such that $f|_V = g|_V$. By $\varepsilon(p)$ we denote the set of equivalence classes $\varepsilon(p) = \hat{\varepsilon}(p)/\equiv$. The elements $[f]$ of $\varepsilon(p)$ are called the **function germs** at p. By the following operations $+$ and $\cdot$ on $\varepsilon(p)$ we make it into an $\mathbb{R}$-algebra.

i. $[f] + [g] = [f + g]$,
ii. $\lambda \cdot [f] = [\lambda \cdot f]$,
iii. $[f] \cdot [g] = [f \cdot g]$

for all $f, g \in \varepsilon(p)$ and $\lambda \in \mathbb{R}$.

Definition 3.1. A **tangent vector** X_p at $p \in M$ is a map $X_p : \varepsilon(p) \to \mathbb{R}$ such that

i. $X_p(\lambda \cdot f + \mu \cdot g) = \lambda \cdot X_p(f) + \mu \cdot X_p(g)$,
ii. $X_p(f \cdot g) = g(p) \cdot X_p(f) + f(p) \cdot X_p(g)$

for all $\lambda, \mu \in \mathbb{R}$ and $f, g \in \varepsilon(p)$. By T_pM we denote the set of all tangent vectors X_p at $p \in M$. T_pM is called the **tangent space** of M at p.

The following operations $+$ and $\cdot$ make the tangent space T_pM into a real vector space.

i. $(X_p + Y_p)(f) = X_p(f) + Y_p(f)$,

ii. $(\lambda \cdot X_p)(f) = \lambda \cdot X_p(f)$

for all $X_p, Y_p \in T_pM$, $f \in \varepsilon(p)$ and $\lambda \in \mathbb{R}$.

For $M = \mathbb{R}^m$ we denote by ε_m the set of function germs at $0 \in \mathbb{R}^m$ i.e. $\varepsilon_m = \varepsilon(0)$. For $v \in \mathbb{R}^m$ and $f \in \varepsilon_m$ the **directional derivative** of f at 0 in the direction of v is given by

$$\partial_v f = \lim_{t \to 0} \frac{f(tv) - f(0)}{t}.$$

It is well known that for $v = (v_1, \dots, v_m)$ we have $\partial_v f = \sum_{i=1}^m v_i \frac{\partial f}{\partial x_i}(0)$ and that

i. $\partial_v(\lambda \cdot f + \mu \cdot g) = \lambda \cdot \partial_v f + \mu \cdot \partial_v g$,
ii. $\partial_v(f \cdot g) = g(0) \cdot \partial_v f + f(0) \cdot \partial_v g$,
iii. $\partial_{(\lambda \cdot v + \mu \cdot w)} f = \lambda \cdot \partial_v f + \mu \cdot \partial_w f$

for all $\lambda, \mu \in \mathbb{R}$, $v, w \in \mathbb{R}^m$ and $f, g \in \varepsilon_m$

Corollary 3.2. *If $v \in \mathbb{R}^m$ then the directional derivative ∂_v is an element of the tangent space $T_0\mathbb{R}^m$.*

Lemma 3.3. *If $f \in \varepsilon_m$ then there exist functions $\psi_k \in \varepsilon_m$ such that*

$$f(x) = f(0) + \sum_{k=1}^m x_k \cdot \psi_k(x) \quad \textit{and} \quad \psi_k(0) = \frac{\partial f}{\partial x_k}(0).$$

PROOF. It follows from the Fundamental Theorem of Calculus that

$$\begin{aligned} f(x) - f(0) &= \int_0^1 \frac{\partial f}{\partial t}(tx_1, \dots, tx_m) dt \\ &= \sum_{k=1}^m x_k \int_0^1 \frac{\partial f}{\partial x_k}(tx_1, \dots, tx_m) dt. \end{aligned}$$

Put $\psi_k(x) = \int_0^1 \partial f / \partial x_k(tx_1, \dots, tx_m) dt$ and the statement immediately follows. □

Theorem 3.4. *The map $\Phi : \mathbb{R}^m \to T_0\mathbb{R}^m$ given by $v \mapsto \partial_v$ is a vector space isomorphism.*

PROOF. That the map Φ is linear follows directly from

$$\partial_{(\lambda \cdot v + \mu \cdot w)} f = \lambda \cdot \partial_v f + \mu \cdot \partial_w f$$

for all $\lambda, \mu \in \mathbb{R}$, $v, w \in \mathbb{R}^m$ and $f \in \varepsilon_m$.

Let $v, w \in \mathbb{R}^m$ such that $v \neq w$. Choose an element $u \in \mathbb{R}^m$ such that $\langle u, v \rangle \neq \langle u, w \rangle$ and define $f : \mathbb{R}^m \to \mathbb{R}$ by $f(x) = \langle u, x \rangle$. Then $\partial_v f = \langle u, v \rangle \neq \langle u, w \rangle = \partial_w f$ so $\partial_v \neq \partial_w$. This proves that the map Φ is injective.

Let α be an arbitrary element of $T_0\mathbb{R}^m$. For $k = 1, \dots, m$ let $\hat{x}_k : \mathbb{R}^m \to \mathbb{R}$ be the map $(x_1, \dots, x_m) \mapsto x_k$ and put $v_k = \alpha(\hat{x}_k)$. For the constant function $1 : (x_1, \dots, x_m) \mapsto 1$ we have $\alpha(1) = \alpha(1 \cdot 1) = 1 \cdot \alpha(1) + 1 \cdot \alpha(1) = 2 \cdot \alpha(1)$, so $\alpha(1) = 0$. By the linearity of α it follows that $\alpha(c) = 0$ for all constants $c \in \mathbb{R}$. Let $f \in \varepsilon_m$ and following Lemma 3.3 write

$$f(x) = f(0) + \sum_{k=1}^{m} \hat{x}_k(x) \cdot \psi_k(x),$$

where $\psi_k \in \varepsilon_m$ with $\psi_k(0) = \partial f/\partial x_k(0)$. Then by applying α we obtain that

$$\begin{aligned}
\alpha(f) &= \alpha(f(0) + \sum_{k=1}^{m} \hat{x}_k \cdot \psi_k) \\
&= \alpha(f(0)) + \sum_{k=1}^{m} \psi_k(0) \cdot \alpha(\hat{x}_k) + \sum_{k=1}^{m} \hat{x}_k(0) \cdot \alpha(\psi_k) \\
&= \sum_{k=1}^{m} v_k \frac{\partial f}{\partial x_k}(0) \\
&= \partial_v f,
\end{aligned}$$

where $v = (v_1, \dots, v_m) \in \mathbb{R}^m$. This implies that $\alpha = \partial_v$ and proves that the map Φ is surjective. $\square$

Corollary 3.5. *Let $\{e_k | k = 1, \dots, m\}$ be a basis for $\mathbb{R}^m$. Then the set $\{\partial_{e_k} | k = 1, \dots, m\}$ is a basis for the tangent space $T_0\mathbb{R}^m$.*

Definition 3.6. Let $\phi : M \to N$ be a map between two manifolds. For a point $p \in M$ we define the map $d\phi_p : T_pM \to T_{\phi(p)}N$ by

$$(d\phi_p)(X_p)(f) = X_p(f \circ \phi)$$

for all $X_p \in T_pM$ and $f \in \varepsilon(\phi(p))$. The map $d\phi_p$ is called the **differential** of ϕ at $p \in M$.

Proposition 3.7. *Let $\phi : M \to \tilde{M}$ and $\psi : \tilde{M} \to N$ be maps between manifolds, then*

i. *the map $d\phi_p : T_pM \to T_{\phi(p)}\tilde{M}$ is linear,*
ii. *if id_M is the identity map, then $d(id_M)_p = id_{T_pM}$,*
iii. *$d(\psi \circ \phi)_p = d\psi_{\phi(p)} \circ d\phi_p$*

for all $p \in M$. The last equation is called the **Chain Rule**.

PROOF. The only point which is not trivial is the chain rule. If $X_p \in T_pM$ and $f \in \varepsilon(\psi \circ \phi(p))$, then

$$\begin{aligned}(d\psi_{\phi(p)} \circ d\phi_p(X_p))(f) &= (d\phi_p(X_p))(f \circ \psi) \\ &= X_p(f \circ \psi \circ \phi) \\ &= d(\psi \circ \phi)_p(X_p)(f).\end{aligned}$$

This proves the statement. □

Corollary 3.8. *Let $\phi : M \to N$ be a diffeomorphism with inverse $\psi = \phi^{-1} : N \to M$. Then the differential $d\phi_p : T_pM \to T_{\phi(p)}N$ at p is bijective and $(d\phi_p)^{-1} = d\psi_{\phi(p)}$.*

PROOF. The statement follows directly from

$$d\phi_p \circ d\psi_{\phi(p)} = d(\phi \circ \psi)_{\phi(p)} = d(\mathrm{id}_N)_{\phi(p)} = \mathrm{id}_{T_{\phi(p)}N},$$
$$d\psi_{\phi(p)} \circ d\phi_p = d(\psi \circ \phi)_p = d(\mathrm{id}_M)_p = \mathrm{id}_{T_pM}.$$

□

As a direct consequence of Corollaries 3.5 and 3.8 we obtain the following result which generalizes the case when M^2 is a surface in $\mathbb{R}^3$.

Corollary 3.9. *Let M^m be an m-dimensional manifold and $p \in M$. Then the tangent space T_pM at p is an m-dimensional real vector space.*

PROOF. Let (U, y) be a chart on M with $y(p) = 0$. Then the linear map $dy_p : T_pM \to T_0\mathbb{R}^m \cong \mathbb{R}^m$ is a vector space isomorphism. □

Next we show that a local chart around a point $p \in M$ gives a canonical basis for the tangent space T_pM.

Proposition 3.10. *Let M^m be a manifold and (U, y) be a local coordinate on M. Further let $\{e_k |\ k = 1, \dots, m\}$ be the canonical basis for $\mathbb{R}^m$. For $p \in M$ we define $\frac{\partial}{\partial y_k}|_p \in T_pM$ by*

$$\frac{\partial}{\partial y_k}|_p : f \mapsto \frac{\partial f}{\partial y_k}(p) = \partial_{e_k}(f \circ y^{-1})(y(p)).$$

Then $\{\frac{\partial}{\partial y_k}|_p \mid k = 1, 2, \dots, m\}$ is a basis for the tangent space T_pM for all $p \in U$.

PROOF. We are assuming as usual that the manifold M is smooth so the inverse y^{-1} of y is smooth with differential $(dy^{-1})_{y(p)} : T_{y(p)}\mathbb{R}^m \to T_pM$ satisfying

$$\begin{aligned}(dy^{-1})_{y(p)}(\partial_{e_k})(f) &= \partial_{e_k}(f \circ y^{-1})(y(p)) \\ &= (\frac{\partial}{\partial y_k}|_p)(f)\end{aligned}$$

for all $f \in \varepsilon(p)$. This proves the statement. $\square$

We shall now give an alternative description of the tangent space T_pM. Let $\hat{C}(p)$ be the set of all locally defined C^1-curves passing through the point $p \in M$ i.e.

$$\hat{C}(p) = \{\gamma : (-\epsilon, \epsilon) \to M |\ \gamma \text{ is } C^1 \text{ and } \gamma(0) = p\}.$$

For an element $\gamma \in \hat{C}(p)$ we have the differential $d\gamma_0 : T_0\mathbb{R} \to T_pM$. Let $e_1 \cong \partial_{e_1} \in \mathbb{R} \cong T_0\mathbb{R}$ be the positive unit tangent of $T_0\mathbb{R}$. On the set $\hat{C}(p)$ we define the equivalence relation $\equiv$ by

$$\gamma_1 \equiv \gamma_2 \text{ if and only if } (d\gamma_1)_0(e_1) = (d\gamma_2)_0(e_1).$$

This means that the two parametrized curves γ_1 and γ_2 are identified if they have the same tangent at the point p. By $C(p)$ we denote the set of equivalence classes i.e. $C(p) = \hat{C}(p)/\equiv$.

It is an easy exercise to show that the map $\Phi : C(p) \to T_pM$ with $\Phi : [\gamma] \mapsto (d\gamma)_0(e_1)$ is bijective. This implies that T_pM can be identified with $C(p)$ being the set of all possible tangents to curves going through the point p. Hence a vector $v \in T_pM$ can be thought of as a first order differential operator acting on the functions defined locally around the point $p \in M$ as follows: Let $f : U \to \mathbb{R}$ be a function defined on an open subset U of M containing p. Furthermore let $\gamma : I \to U$ be a curve with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Then the action of v on f is given by

$$v(f) = \frac{d}{dt}(f \circ \gamma(t))|_{t=0}.$$

Note that this is independent of the choice of the curve γ as long as $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.

This second interpretation of T_pM shows that if $m < n$ and M^m is a submanifold of $\mathbb{R}^n$, then T_pM is simply the tangent space of M at p in the classical sense i.e. the space of all tangents to curves at the point $p \in M$. We can now use this alternative description to determine the tangent space $T_e\mathbf{O}(m)$ of the orthogonal group $\mathbf{O}(m)$ at the neutral element e.

Example 3.11. Let e denote the neutral element of the orthogonal group $\mathbf{O}(m)$ i.e. the identity matrix. Let $A : (-\epsilon, \epsilon) \to \mathbf{O}(m)$ be a curve in $\mathbf{O}(m)$ such that $A(0) = e$. Then $A(s) \cdot A(s)^t = e$ for all $s \in (-\epsilon, \epsilon)$. Differentiation yields

$$\{A'(s) \cdot A(s)^t + A(s) \cdot A'(s)^t\}|_{s=0} = 0$$

or equivalently $A'(0) + A'(0)^t = 0$. This means that each tangent vector of $\mathbf{O}(m)$ at e is a skew-symmetric matrix.

On the other hand, for an arbitrary real skew-symmetric matrix Z define $\gamma : \mathbb{R} \to \mathbb{R}^{m\times m}$ by $\gamma : s \mapsto \mathrm{Exp}(s \cdot Z)$, where Exp is the usual exponential map for matrices defined in Exercise 3.1. Then

$$\begin{aligned}\gamma(s)\cdot\gamma(s)^t &= \mathrm{Exp}(s\cdot Z)\cdot\mathrm{Exp}(s\cdot Z)^t \\ &= \mathrm{Exp}(s\cdot Z)\cdot\mathrm{Exp}(s\cdot Z^t) \\ &= \mathrm{Exp}(s(Z+Z^t)) \\ &= \mathrm{Exp}(0) \\ &= e.\end{aligned}$$

This shows that γ is a curve on the orthogonal group, $\gamma(0) = e$ and $\gamma'(0) = Z$ so Z is an element of $T_e\mathbf{O}(m)$. Hence

$$T_e\mathbf{O}(m) = \{X \in \mathbb{R}^{m\times m} \mid X + X^t = 0\}.$$

The dimension of $T_e\mathbf{O}(m)$ is therefore $m(m-1)/2$. This confirms our calculations of the dimension of $\mathbf{O}(m)$ in chapter 2 since we know that $\dim(\mathbf{O}(m)) = \dim(T_e\mathbf{O}(m))$.

Exercises

Exercise 3.1. Let the exponential map $\text{Exp} : \mathbb{C}^{m\times m} \to \mathbb{C}^{m\times m}$ be defined by

$$\text{Exp} : A \mapsto \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

Prove that for all $A, B \in \mathbb{C}^{m\times m}$

i. $\det[\text{Exp}(A)] = e^{\text{trace}(A)}$,
ii. if $A \cdot B = B \cdot A$ then $\text{Exp}(A + B) = \text{Exp}(A) \cdot \text{Exp}(B)$,
iii. $\text{Exp}(\bar{A}^t) = \overline{\text{Exp}(A)^t}$.

Exercise 3.2. Use the results from Exercise 3.1 to determine the tangent space T_eG at the neutral element e and the dimension of the following Lie groups: $\mathbf{GL}(\mathbb{R}^m)$, $\mathbf{SL}(\mathbb{R}^m)$, $\mathbf{O}(m)$, $\mathbf{SO}(m)$, $\mathbf{GL}(\mathbb{C}^m)$, $\mathbf{SL}(\mathbb{C}^m)$, $\mathbf{U}(m)$, $\mathbf{SU}(m)$.

Exercise 3.3. Let p be an arbitrary point on the unit sphere S^{2n+1} of $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$. Determine the tangent space T_pS^{2n+1} and show that it contains an n-dimensional complex subspace of $\mathbb{C}^{n+1}$.

CHAPTER 4

The Tangent Bundle

In this chapter we construct for each differentiable manifold $(M, \hat{\mathcal{A}})$ its tangent bundle TM. Intuitively this is the object we get by glueing at each point $p \in M$ the corresponding tangent space T_pM. This way we obtain a $2m$-dimensional topological manifold. The structure $\hat{\mathcal{A}}$ on M induces a differentiable structure $\widehat{\mathcal{A}^*}$ on TM which makes $(TM, \widehat{\mathcal{A}^*})$ into a differentiable manifold. We then study vector fields which can be thought of as being maps from M into TM. These are fundamental tools for the geometric study of manifolds.

Definition 4.1. Let E and M be topological manifolds and $\pi : E \to M$ be a continuous surjective map. The triple (E, M, π) is called an n-dimensional **topological vector bundle** over M if

i. for each $p \in M$ the fibre $E_p = \pi^{-1}(p)$ is an n-dimensional vector space,
ii. for each $p \in M$ there exists a **bundle chart** $(\pi^{-1}(U), \psi)$ consisting of the pre-image $\pi^{-1}(U)$ of an open neighbourhood U of p and a homeomorphism $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that for all $q \in U$ the map $\psi_q = \psi|_{E_q} : E_q \to \{q\} \times \mathbb{R}^n$ is a vector space isomorphism.

Definition 4.2. Let (E, M, π) be an n-dimensional topological vector bundle over M. It is said to be **trivial** if there exists a global bundle chart $\psi : E \to M \times \mathbb{R}^n$.

Example 4.3. Let M be a topological manifold and $\pi : M \times \mathbb{R}^n \to M$ be the natural projection $\pi : (x, v) \mapsto x$. Then $(M \times \mathbb{R}^n, M, \pi)$ is a trivial n-dimensional vector bundle.

Definition 4.4. Let (E, M, π) be a topological vector bundle. A continuous map $\sigma : M \to E$ is called a **section** of the bundle if $\pi \circ \sigma(p) = p$ for each $p \in M$.

Definition 4.5. Let (E, M, π) be an n-dimensional topological vector bundle over M. A collection

$$\mathcal{B} = \{(\pi^{-1}(U_\alpha), \psi_\alpha)|\ \alpha \in I\}$$

of bundle charts is called a **bundle atlas** for (E, M, π) if $M = \cup_\alpha U_\alpha$. For each pair (α, β) there exist a function $A_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathbf{GL}(\mathbb{R}^n)$ such that the corresponding continuous map

$$\psi_\beta \circ \psi_\alpha^{-1}|_{(U_\alpha \cap U_\beta)\times\mathbb{R}^n} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

is given by

$$(p, v) \mapsto (p, (A_{\alpha,\beta}(p))(v)).$$

The elements of $\{A_{\alpha,\beta} |\ \alpha, \beta \in I\}$ are called the **transition maps** of the bundle atlas $\mathcal{B}$.

Definition 4.6. Let E and M be smooth manifolds and $\pi : E \to M$ be a smooth map such that (E, M, π) is an n-dimensional topological vector bundle. A bundle atlas $\mathcal{B}$ for (E, M, π) is said to be **smooth** if the corresponding transition maps are smooth. A **smooth vector bundle** is a topological vector bundle together with a maximal smooth bundle atlas. A smooth section of (E, M, π) is called a **vector field**. By $C^\infty(E)$ we denote the set of all vector fields of (E, M, π).

From now on we assume that all vector bundles are smooth.

Definition 4.7. Let M be a manifold and (E, M, π) be a vector bundle over M. By the following operations we make $C^\infty(E)$ into a $C^\infty(M, \mathbb{R})$-module. In particular, $C^\infty(E)$ is a vector space over the real numbers.

i. $(v + w)_p = v_p + w_p$,
ii. $(f \cdot v)_p = f(p) \cdot v_p$

for all $v, w \in C^\infty(E)$ and $f \in C^\infty(M, \mathbb{R})$.

Definition 4.8. Let M be a manifold and (E, M, π) be an n-dimensional vector bundle over M. A set $F = \{v_1, \dots, v_n\}$ of vector fields $v_1, \dots, v_n : U \subset M \to E$ is called a **local frame** for E on U if for each $p \in U$ the set $\{(v_1)_p, \dots, (v_n)_p\}$ is a basis for the vector space E_p.

Example 4.9. For a manifold $(M, \hat{\mathcal{A}})$ we denote by TM the **tangent bundle** of M given by

$$TM = \{(p, v) |\ p \in M,\ v \in T_pM\}$$

and define the projection $\pi : TM \to M$ by $\pi : (p, v) \mapsto p$.

For a chart $x : U \to \mathbb{R}^m$ in $\hat{\mathcal{A}}$ we define $x^* : \pi^{-1}(U) \to \mathbb{R}^m \times \mathbb{R}^m$ by

$$x^* : (p, \sum_{k=1}^m v_k \frac{\partial}{\partial x_k}|_p) \mapsto (x(p), (v_1, \dots, v_m)).$$

Then the collection

$$\{(x^*)^{-1}(W) \subset TM |\ (U,x) \in \hat{\mathcal{A}} \ \text{ and } \ W \subset x(U) \times \mathbb{R}^m \ \text{open}\}$$

is a basis for a topology $\mathcal{T}_{TM}$ on TM and $(\pi^{-1}(U), x^*)$ is a chart on the $2m$-dimensional topological manifold $(TM, \mathcal{T}_{TM})$. If (U,x) and (V,y) are two charts in $\hat{\mathcal{A}}$ such that $p \in U \cap V$, then the transition map

$$(y^*) \circ (x^*)^{-1} : x^*(\pi^{-1}(U \cap V)) \to \mathbb{R}^m \times \mathbb{R}^m$$

is given by

$$(a,b) \mapsto (y \circ x^{-1}(a), \sum_{k=1}^m \frac{\partial y_1}{\partial x_k}(x^{-1}(a))b_k, \dots, \sum_{k=1}^m \frac{\partial y_m}{\partial x_k}(x^{-1}(a))b_k).$$

We are assuming that $y \circ x^{-1}$ is smooth so it follows that $(y^*) \circ (x^*)^{-1}$ is also smooth. This means that

$$\mathcal{A}^* = \{(\pi^{-1}(U), x^*) |\ (U,x) \in \hat{\mathcal{A}}\}$$

is a C^∞-atlas on TM so $(TM, \widehat{\mathcal{A}^*})$ has the structure of a **smooth manifold**. It is trivial that the projection $\pi : TM \to M$ is smooth and surjective.

For each point $p \in M$ the fibre $\pi^{-1}(p)$ of π is the tangent space T_pM of M at p so it is an m-dimensional vector space. For a chart $x : U \to \mathbb{R}^m$ in $\hat{\mathcal{A}}$ we define $\bar{x} : \pi^{-1}(U) \to U \times \mathbb{R}^m$ by

$$\bar{x} : (p, \sum_{k=1}^m v_k \frac{\partial}{\partial x_k}|_p) \mapsto (p, (v_1, \dots, v_m)).$$

The restriction $\bar{x}_p = \bar{x}|_{T_pM} : T_pM \to \{p\} \times \mathbb{R}^m$ to T_pM is given by

$$\bar{x}_p : \sum_{k=1}^m v_k \frac{\partial}{\partial x_k}|_p \mapsto (v_1, \dots, v_m),$$

hence a vector space isomorphism. This implies that the map $\bar{x} : \pi^{-1}(U) \to U \times \mathbb{R}^m$ is a bundle chart. It is not difficult to see that

$$\mathcal{B} = \{(\pi^{-1}(U), \bar{x}) |\ (U,x) \in \hat{\mathcal{A}}\}$$

is a bundle atlas making (TM, M, π) into an m-dimensional topological vector bundle. It immediately follows from above that (TM, M, π) together with the maximal bundle atlas $\hat{\mathcal{B}}$ defined by $\mathcal{B}$ is a smooth vector bundle.

Example 4.10. Let S^3 be the unit sphere in $\mathbb{C}^2$ on which we define the operation $\cdot$ by

$$(x,y) \cdot (\alpha, \beta) = (x\alpha - \bar{y}\beta, y\alpha + \bar{x}\beta).$$

It is easily checked that $(S^3, \cdot)$ is a Lie group with neutral element $e = (1,0)$. Put $v_1 = (i,0)$, $v_2 = (0,1)$ and $v_3 = (0,i)$ and for $k = 1,2,3$ define the curves $\gamma_k : \mathbb{R} \to S^3$ by

$$\gamma_k : t \mapsto \cos t \cdot (1,0) + \sin t \cdot v_k.$$

Then $\gamma_k(0) = e$ and $\dot\gamma_k(0) = v_k$ for each k so v_1, v_2 and v_3 are elements of the tangent space T_eS^3. The tangent vectors v_1, v_2 and v_3 are linearily independent so they span T_eS^3. The left translations $L_p : S^3 \to S^3$ with $L_p : h \mapsto p \cdot h$ induce vector fields $X_1, X_2, X_3 \in C^\infty(TS^3)$ by

$$(X_k)_p = (dL_p)_e(v_k) = \frac{d}{dt}(L_p(\gamma_k(t)))|_{t=0}.$$

It is left as an exercise for the reader to show that at a point $p = (x,y) \in S^3$ the values of X_k at p is given by

$$\begin{aligned}(X_1)_p &= (x,y)\cdot(i,0) = (ix, iy),\\ (X_2)_p &= (x,y)\cdot(0,1) = (-\bar y, \bar x),\\ (X_3)_p &= (x,y)\cdot(0,i) = (-i\bar y, i\bar x).\end{aligned}$$

Lemma 4.11. *Let M^m be a smooth manifold and $X : M \to TM$ be a continuous section on M. Then the following conditions are equivalent*

i. *the section X is smooth,*

ii. *if (U,x) is a chart on M then the functions $a_1, \dots, a_m : U \to \mathbb{R}$ given by*

$$\sum_{k=1}^m a_k \frac{\partial}{\partial x_k} = X|_U,$$

are smooth,

iii. *if $f : V \to \mathbb{R}$ defined on an open subset V of M is smooth, then the function $X(f) : V \to \mathbb{R}$ with $X(f)(p) = X_p(f)$ is smooth.*

PROOF. *i.* $\Rightarrow$ *ii*: The functions $a_k = \pi_{m+k} \circ x^* \circ X|_U : U \to TM \to x(U) \times \mathbb{R}^m \to \mathbb{R}$ are restrictions of compositions of smooth maps so therefore smooth.

ii. $\Rightarrow$ *iii*: Let (U,x) be a chart on M such that U is contained in V. By assumption the map $X(f|_U) = \sum_{i=1}^m a_i \frac{\partial f}{\partial x_i}$ is smooth. This is true for each such chart (U,x) so the function $X(f)$ is smooth.

iii. $\Rightarrow$ *i*: Note that the smoothness of the section X is equivalent to $x^* \circ X|_U : U \to \mathbb{R}^{2m}$ being smooth for all charts (U,x) on M. On the other hand, this is equivalent to $x_k^* = \pi_k \circ x^* \circ X|_U : U \to \mathbb{R}$ being smooth for all $k = 1,2,\dots,2m$ and all charts (U,x) on M. It is trivial that the coordinates $x_k^* = x_k$ for $k = 1,\dots,m$ are smooth. But $x_{m+k}^* = a_k = X(x_k)$ for $k = 1,\dots,m$ hence also smooth by assumption. □

Definition 4.12. Let M be a smooth manifold. For two vector fields $X, Y \in C^\infty(TM)$ we define the **Lie bracket** $[X,Y]_p$ of X and Y at $p \in M$ by

$$[X,Y]_p(f) = X_p(Y(f)) - Y_p(X(f)) \in \mathbb{R}$$

where $f \in C^\infty(M,\mathbb{R})$.

Lemma 4.13. *Let M be a smooth manifold, $X, Y \in C^\infty(TM)$ be vector fields on M, $f, g \in C^\infty(M,\mathbb{R})$ and $\lambda, \mu \in \mathbb{R}$. Then*

i. $[X,Y]_p(\lambda f + \mu g) = \lambda[X,Y]_p(f) + \mu[X,Y]_p(g)$,
ii. $[X,Y]_p(f \cdot g) = f(p)[X,Y]_p(g) + g(p)[X,Y]_p(f)$.

PROOF.

$$\begin{aligned}
& [X,Y]_p(\lambda f + \mu g) \\
= \ & X_p(Y(\lambda f + \mu g)) - Y_p(X(\lambda f + \mu g)) \\
= \ & \lambda X_p(Y(f)) + \mu X_p(Y(g)) - \lambda Y_p(X(f)) - \mu Y_p(X(g)) \\
= \ & \lambda[X,Y]_p(f) + \mu[X,Y]_p(g).
\end{aligned}$$

$$\begin{aligned}
& [X,Y]_p(f \cdot g) \\
= \ & X_p(Y(f \cdot g)) - Y_p(X(f \cdot g)) \\
= \ & X_p(f \cdot Y(g) + g \cdot Y(f)) - Y_p(f \cdot X(g) + g \cdot X(f)) \\
= \ & X_p(f)Y_p(g) + f(p)X_p(Y(g)) + X_p(g)Y_p(f) + g(p)X_p(Y(f)) \\
& -Y_p(f)X_p(g) - f(p)Y_p(X(g)) - Y_p(g)X_p(f) - g(p)Y_p(X(f)) \\
= \ & f(p)\{X_p(Y(g)) - Y_p(X(g))\} + g(p)\{X_p(Y(f)) - Y_p(X(f))\} \\
= \ & f(p)[X,Y]_p(g) + g(p)[X,Y]_p(f).
\end{aligned}$$

□

Proposition 4.14. *Let M be a manifold and $X, Y \in C^\infty(TM)$. Then*

i. *$[X,Y]_p$ is an element of T_pM for all $p \in M$,*
ii. *the section $[X,Y] : p \mapsto [X,Y]_p$ is smooth.*

PROOF. The first statement is a direct consequence of Lemma 4.13. It implies that $[X,Y] : M \to TM$ is a section of TM. If $f : M \to \mathbb{R}$ is a smooth function, then $[X,Y](f) = X(Y(f)) - Y(X(f))$ is smooth. It then follows from Lemma 4.11 that the section $[X,Y]$ is smooth. □

Theorem 4.15. *Let M be a smooth manifold. The vector space $C^\infty(TM)$ of smooth vector fields on M equipped with the Lie bracket $[,] : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$ is a* **Lie algebra** *over the real numbers i.e. if $X, Y, Z \in C^\infty(TM)$ and $\lambda, \mu \in \mathbb{R}$ then*

i. $[\lambda X + \mu Y, Z] = \lambda[X,Z] + \mu[Y,Z]$,

ii. $[X, Y] = -[Y, X]$,
iii. $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$.

PROOF. See exercise 4.4. □

Definition 4.16. Let M be a smooth manifold. Two vector fields $X, Y \in C^\infty(TM)$ are said to **commute** if $[X, Y] = 0$.

Lemma 4.17. *Let $\phi : M \to N$ be a smooth bijective map between two manifolds. If $X, Y \in C^\infty(TM)$ are vector fields on M, then*

i. $d\phi(X) \in C^\infty(TN)$,
ii. *the map $d\phi : C^\infty(TM) \to C^\infty(TN)$ is a Lie algebra homomorphism i.e.* $[d\phi(X), d\phi(Y)] = d\phi([X, Y])$.

PROOF. That $d\phi(X) \in C^\infty(TN)$ follows directly from the fact that

$$d\phi(X)(f)(\phi(p)) = X(f \circ \phi)(p).$$

Let $f : N \to \mathbb{R}$ be a smooth function, then

$$\begin{aligned}
[d\phi(X), d\phi(Y)](f) &= d\phi(X)(d\phi(Y)(f)) - d\phi(Y)(d\phi(X)(f)) \\
&= X(d\phi(Y)(f) \circ \phi) - Y(d\phi(X)(f) \circ \phi) \\
&= X(Y(f \circ \phi)) - Y(X(f \circ \phi)) \\
&= [X, Y](f \circ \phi) \\
&= d\phi([X, Y])(f).
\end{aligned}$$

This completes the proof. □

Definition 4.18. Let G be a Lie group with neutral element e. For $x \in G$ the **left translation** by x is the map $L_x : G \to G$ defined by $L_x : y \mapsto xy$. A vector field $X \in C^\infty(TG)$ is said to be **left-invariant** if for all $x, y \in G$

$$X_{xy} = (dL_x)_y(X_y).$$

We denote the set of all left-invariant vector fields on G by $\mathfrak{g}$.

Note that if $X \in \mathfrak{g}$, then $X_x = (dL_x)_e(X_e)$. This implies that the value X_x of X at $x \in G$ is completely determined by the value X_e of X at e.

Proposition 4.19. *Let G be a Lie group and $\mathfrak{g}$ be the set of all left-invariant vector fields on G. Then*

i. $\mathfrak{g}$ *is a Lie subalgebra of* $C^\infty(TG)$ *i.e. if* $X, Y \in \mathfrak{g}$ *then* $[X, Y] \in \mathfrak{g}$,
ii. *the vector spaces $\mathfrak{g}$ and T_eG are isomorphic.*

PROOF. If $x, y \in G$ and f a smooth function on G, then

$$\begin{aligned}((dL_x)_y[X,Y]_y)(f) &= [dL_x(X), dL_x(Y)]_y(f) \\ &= dL_x(X)_y(dL_x(Y)(f)) - dL_x(Y)_y(dL_x(X)(f)) \\ &= dL_x(X)_y(Y(f \circ L_x)) - dL_x(Y)_y(X(f \circ L_x)) \\ &= X(Y(f \circ L_x))(y) - Y(X(f \circ L_x))(y) \\ &= [X,Y]_{xy}(f)\end{aligned}$$

for all $X, Y \in \mathfrak{g}$. This proves that $[X,Y] \in \mathfrak{g}$ and thereby that the vector space $\mathfrak{g}$ is a Lie subalgebra of $C^\infty(TG)$.

To see that $\mathfrak{g}$ is isomorphic to the tangent space at the neutral element T_eG note that the map $* : T_eG \to \mathfrak{g}$ given by $* : Z \mapsto (Z^* : x \mapsto (dL_x)_e(Z))$ is a vector space isomorphism. □

Definition 4.20. Let G be a Lie group and define a Lie bracket $[,]$ on T_eG by $[X,Y] = [X^*, Y^*]_e$. Then $(T_eG, [,]) \cong (\mathfrak{g}, [,])$ is called the **Lie algebra** of G.

Proposition 4.21. *Let $\mathbf{GL}(\mathbb{R}^m)$ be the general linear group of the real m-dimensional vector space $\mathbb{R}^m$. Then $T_e\mathbf{GL}(\mathbb{R}^m) = \mathbb{R}^{m\times m}$ and the Lie bracket $[,] : T_e\mathbf{GL}(\mathbb{R}^m) \times T_e\mathbf{GL}(\mathbb{R}^m) \to T_e\mathbf{GL}(\mathbb{R}^m)$ defined above is given by $[A,B] = AB - BA$.*

PROOF. See exercise 4.6. □

The Lie algebras of the matrix groups introduced in Example 2.23 are denoted by $\mathfrak{gl}(\mathbb{R}^m)$, $\mathfrak{sl}(\mathbb{R}^m)$, $\mathfrak{o}(m)$, $\mathfrak{so}(m)$, $\mathfrak{gl}(\mathbb{C}^m)$, $\mathfrak{sl}(\mathbb{R}^m)$, $\mathfrak{u}(m)$ and $\mathfrak{su}(m)$.

Theorem 4.22. *Let G be a Lie group. Then the tangent bundle TG is trivial.*

PROOF. Let $\{X_1, \dots, X_m\}$ be a basis for T_eG. Then the map $\psi : TG \to G \times \mathbb{R}^m$ given by

$$\psi : (p, \sum_{k=1}^m v_k \cdot (X_k^*)_p) \mapsto (p, (v_1, \dots, v_m))$$

is a global bundle chart so the tangent bundle TG is trivial. □

Exercises

Exercise 4.1. Let $(M^m, \hat{\mathcal{A}})$ be a smooth manifold and (U, x), (V, y) be two charts in $\hat{\mathcal{A}}$ such that $U \cap V \neq \emptyset$. Let $f = y \circ x^{-1} : x(U \cap V) \to \mathbb{R}^m$ be the corresponding transition map. Show that the local frames $\{\frac{\partial}{\partial x_i} |\ i = 1, \dots, m\}$ and $\{\frac{\partial}{\partial y_j} |\ j = 1, \dots, m\}$ for TM on $U \cap V$ are related by

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{m} \frac{\partial (f_j \circ x)}{\partial x_i} \cdot \frac{\partial}{\partial y_j}.$$

Exercise 4.2. Let $\mathbf{O}(m)$ be the orthogonal group.

i. Find a basis for the tangent space $T_e\mathbf{O}(m)$,
ii. construct a non-vanishing vector field $Z \in C^\infty(T\mathbf{O}(m))$,
iii. determine all smooth vector fields on $\mathbf{O}(2)$.

The Hairy Ball Theorem. If $m \in \mathbb{N}^+$ then there does not exist a non-vanishing vector field $X \in C^0(TS^{2m})$.

Exercise 4.3. Let $m \in \mathbb{N}^+$. Use the Hairy Ball Theorem to prove that the tangent bundle TS^{2m} of S^{2m} is not trivial. Construct a non-vanishing vector field $X \in C^\infty(TS^{2m+1})$.

Exercise 4.4. Find a proof for Theorem 4.15.

Exercise 4.5. Let $\{\partial/\partial x_k |\ k = 1, \dots, m\}$ be the standard global frame for $T\mathbb{R}^m$. Let $X, Y \in C^\infty(T\mathbb{R}^m)$ be two vector fields given by

$$X = \sum_{k=1}^{m} \alpha_k \frac{\partial}{\partial x_k} \quad \text{and} \quad Y = \sum_{k=1}^{m} \beta_k \frac{\partial}{\partial x_k},$$

where $\alpha_k, \beta_k \in C^\infty(\mathbb{R}^m, \mathbb{R})$. Calculate the Lie brackets $[\partial/\partial x_k, \partial/\partial x_l]$ and $[X, Y]$.

Exercise 4.6. Find a proof for Proposition 4.21.

CHAPTER 5

Immersions, Embeddings and Submersions

Definition 5.1. A map $\phi : M^m \to N^n$ between two manifolds is called

i. an **immersion** if for each $p \in M$ the differential $d\phi_p : T_pM \to T_{\phi(p)}N$ is injective,
ii. an **embedding** if it is an immersion and a homeomorphism onto its image $\phi(M)$,
iii. a **submersion** if for each $p \in M$ the differential $d\phi_p$ is surjective.

Example 5.2. Let S^1 be the unit circle in $\mathbb{C}$. For each positive natural number k define $\phi_k : S^1 \to \mathbb{C}$ and $\psi_k : S^1 \to S^1$ by

$$\phi_k, \psi_k : z \mapsto z^k.$$

For a point $p \in S^1$ let $\gamma : \mathbb{R} \to S^1$ be the curve with $\gamma : t \mapsto e^{it}p$. Then $\gamma(0) = p$ and $\dot{\gamma}(0) = ip$. For the differentials of ϕ_k and ψ_k we have

$$(d\psi_k)_p(\dot{\gamma}(0)) = (d\phi_k)_p(\dot{\gamma}(0)) = \frac{d}{dt}(\phi_k \circ \gamma(t))|_{t=0} = \frac{d}{dt}(e^{ikt}p^k)|_{t=0} = ikp^k.$$

The differentials $(d\psi_k)_p : T_pS^1 \cong \mathbb{R}^1 \to T_{p^k}S^1 \cong \mathbb{R}$ are all bijective, so the maps ψ_k are both immersions and submersions. The only one that is an embedding is ψ_1.

The differentials $(d\phi_k)_p : T_pS^1 \to T_{p^k}\mathbb{C} \cong \mathbb{R}^2$ are all injective and not surjective. This means that the maps ϕ_k are all immersions, but none of them is a submersion. The only one that is an embedding is ϕ_1.

Example 5.3. For $p \in S^m$ let $\rho_p : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ be the reflection about the line $\{tp \in \mathbb{R}^{m+1} |\ t \in \mathbb{R}\}$ spanned by p. Then define the map $\phi : S^m \to \mathbb{R}^{(m+1)\times(m+1)}$ by

$$\phi : p \mapsto (\rho_p : q \mapsto 2\langle q, p\rangle p - q).$$

Then the matrix describing the linear map ρ_p is given by $(2pp^t - I) \in \mathbb{R}^{(m+1)\times(m+1)}$, since $2\langle q, p\rangle p - q = 2p\langle p, q\rangle - q = (2pp^t - I)q$

Proposition 5.4. *The map ϕ is an immersion and the image $\phi(S^m)$ is diffeomorphic to the m-dimensional real projective space $\mathbb{R}P^m$ and lies in $Sym(\mathbb{R}^{m+1}) \cap \mathbf{O}(m+1)$.*

Proof. Let p be an arbitrary point on S^m and $\alpha, \beta : I \to S^m$ be two curves meeting at p, that is $\alpha(0) = p = \beta(0)$, with $a = \dot{\alpha}(0)$ and $b = \dot{\beta}(0)$. For $\gamma \in \{\alpha, \beta\}$ we have

$$\phi \circ \gamma : t \mapsto (q \mapsto 2\langle q, \gamma(t)\rangle \gamma(t) - q)$$

so

$$(d\phi)_p(\dot{\gamma}(0)) = \frac{d}{dt}(\phi \circ \gamma(t))|_{t=0} = (q \mapsto 2\langle q, \dot{\gamma}(0)\rangle \gamma(0) + 2\langle q, \gamma(0)\rangle \dot{\gamma}(0)).$$

This means that

$$d\phi_p(a) = (q \mapsto 2\langle q, a\rangle p + 2\langle q, p\rangle a)$$

$$d\phi_p(b) = (q \mapsto 2\langle q, b\rangle p + 2\langle q, p\rangle b).$$

If $a \neq b$ then $d\phi_p(a) \neq d\phi_p(b)$ so the differential $d\phi_p$ is injective. This proves that ϕ is an immersion.

If two points $p, q \in S^m$ are linearly independent, then $\rho_p \neq \rho_q$ since their images are different. On the other hand, if $p = \pm q$ then $\rho_p = \rho_q$. This means that the image $\phi(S^m)$ is diffeomorphic to the quotient space $S^m/\equiv$ where $\equiv$ is the equivalence relation defined by $x \equiv y$ if and only if $x = \pm y$. This proves that $\phi(S^m) \cong \mathbb{R}P^m$. It is obvious that $\phi(S^m)$ lies on $\text{Sym}(\mathbb{R}^{m+1}) \cap \mathbf{O}(m+1)$. □

Corollary 5.5. *The map* $\hat{\phi} : \mathbb{R}P^m \to Sym(\mathbb{R}^{m+1}) \cap \mathbf{O}(m+1)$ *given by*

$$\hat{\phi} : [p] \mapsto (2pp^t - I)$$

is an embedding.

The following result was proved by H. Whitney in his very famous paper, *Differentiable Manifolds*, Ann. of Math. **37** (1936), 645-680.

Deep Result 5.6. *For* $1 \leq r \leq \infty$ *let* M^m *be an* m*-dimensional* C^r*-manifold. Then there exists a* C^r*-embedding* $\phi : M \to \mathbb{R}^{2m+1}$ *into the* $(2m+1)$*-dimensional real vector space* $\mathbb{R}^{2m+1}$*.*

Definition 5.7. Let $\phi : M^m \to N^n$ be a map between manifolds. A point $p \in M$ is called a **critical point** if the differential $d\phi_p : T_pM \to T_{\phi(p)}N$ is not of full rank, and a **regular point** if it is not critical. A point $q \in \phi(M)$ is called a **regular value** if every point on the pre-image $\phi^{-1}(\{q\})$ is regular.

Theorem 5.8 (The Implicit Function Theorem)**.** *Let* $\phi : M^m \to N^n$ *be a map between two manifolds. If* $q \in \phi(M)$ *is a regular value, then the pre-image* $\phi^{-1}(\{q\})$ *is an* $(m-n)$*-dimensional submanifold of* M^m*. The tangent space* $T_p\phi^{-1}(\{q\})$ *of* $\phi^{-1}(\{q\})$ *at* p *is the kernel of the differential* $d\phi_p$ *i.e.* $T_p\phi^{-1}(\{q\}) = Ker\ d\phi_p$*.*

PROOF. Let (V_q, ψ_q) be a chart on N with $q \in V_q$ and $\psi_q(q) = 0$. For a point $p \in \phi^{-1}(\{q\})$ we choose a chart (U_p, ψ_p) on M such that $p \in U_p$, $\psi_p(p) = 0$ and $\phi(U_p) \subset V_q$. The differential of the map

$$\hat{\phi} = \psi_q \circ \phi \circ \psi_p^{-1}|_{\psi_p(U_p)} : \psi_p(U_p) \to \mathbb{R}^n$$

at the point 0 is given by

$$d\hat{\phi}_0 = (d\psi_q)_q \circ d\phi_p \circ (d\psi_p^{-1})_0 : T_0\mathbb{R}^m \to T_0\mathbb{R}^n.$$

The pairs (U_p, ψ_p) and (V_q, ψ_q) are charts so the differentials $(d\psi_q)_q$ and $(d\psi_p^{-1})_0$ are bijective. This means that the differential $d\hat{\phi}_0$ is surjective since $d\phi_p$ is. It then follows from the implicit function theorem 2.13 that $\psi_p(\phi^{-1}(\{q\}) \cap U_p)$ is an $(m-n)$-dimensional submanifold of $\psi_p(U_p)$. Hence $\phi^{-1}(\{q\}) \cap U_p$ is an $(m-n)$-dimensional submanifold of U_p. This is true for each point $p \in \phi^{-1}(\{q\})$ so we have proved that $\phi^{-1}(\{q\})$ is an $(m-n)$-dimensional submanifold of M^m.

Let $\gamma : (-\epsilon, \epsilon) \to \phi^{-1}(\{q\})$ be a curve, such that $\gamma(0) = p$. Then

$$(d\phi)_p(\dot{\gamma}(0)) = \frac{d}{dt}(\phi \circ \gamma(t))|_{t=0} = \frac{dq}{dt}|_{t=0} = 0.$$

This implies that $T_p\phi^{-1}(\{q\})$ is contained in, and has the same dimension as the kernel of $d\phi_p$, so $T_p\phi^{-1}(\{q\}) = \text{Ker } d\phi_p$. □

Definition 5.9. Let M^m be a smooth manifold and U be an open subset of $\mathbb{R}^m$. An immersion $\psi : U \to M$ is called a local **parametrization** of M.

Example 5.10. Let (U, ϕ) be a chart on M^m. Then the inverse $\phi^{-1} : \phi(U) \to U$ of ϕ is a local parametrization of $U \subset M$.

Example 5.11. Let S^3 and S^2 be the unit spheres in $\mathbb{C}^2$ and $\mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$, repectively. Let $\phi : S^3 \to S^2$ be the map given by $\phi : (x, y) \mapsto (2x\bar{y}, |x|^2 - |y|^2)$. Then one easily shows that ϕ and $d\phi_p : T_pS^3 \to T_{\phi(p)}S^2$ are surjective for each $p \in S^3$. This implies that each point $q \in S^2$ is a regular value and the fibres of ϕ are 1-dimensional submanifolds of S^3. They are the great circles given by

$$\phi^{-1}(\{(2x\bar{y}, |x|^2 - |y|^2)\}) = \{e^{i\theta}(x, y) | \ \theta \in \mathbb{R}\}.$$

This means that S^3 is the union of disjoint great circles

$$S^3 = \bigcup_{q \in S^2} \phi^{-1}(\{q\}).$$

It is also what is called a S^1-bundle over S^2.

Exercises

Exercise 5.1. For each $k \in \mathbb{N}_0$ define $\phi_k : \mathbb{C} \to \mathbb{C}$ and $\psi_k : \mathbb{C}^* \to \mathbb{C}$ by $\phi_k, \psi_k : z \mapsto z^k$. For which $k \in \mathbb{N}_0$ are ϕ_k, ψ_k immersions, submersions or embeddings.

Exercise 5.2. Let S^2 and S^3 be the unit spheres of $\mathbb{R}^3$ and $\mathbb{C}^2$, respectively. The **Hopf-map** $\phi : S^3 \to S^2$ is defined by $\phi : (x, y) \mapsto (2x\bar{y}, |x|^2 - |y|^2)$. Prove that ϕ is a submersion.

CHAPTER 6

Riemannian Manifolds

Let M be a smooth manifold and as before we denote by $C^\infty(TM)$ the set of vector fields on M. Let $C_0^\infty(TM) = C^\infty(M, \mathbb{R})$ be the ring of all smooth functions defined on M. For $k \in \mathbb{N}^+$ let

$$C_k^\infty(TM) = \bigotimes_{l=1}^{k} C^\infty(TM),$$

be the k-fold tensor product of $C^\infty(TM)$. Then $C_k^\infty(TM)$ is a $C_0^\infty(TM)$ module in the trivial way. A **tensor field** B **on** M **of type** (r, s) is an r-linear map $B : C_r^\infty(TM) \to C_s^\infty(TM)$ over the ring $C_0^\infty(TM)$ i.e.

$$\begin{aligned}&B(X_1, \dots, X_{l-1}, f \cdot X_l + g \cdot Y, X_{l+1}, \dots, X_r)\\&= f \cdot B(X_1, \dots, X_r) + g \cdot B(X_1, \dots, X_{l-1}, Y, X_{l+1}, \dots, X_r)\end{aligned}$$

for all $X_1, \dots, X_r, Y \in C^\infty(TM)$, $f, g \in C_0^\infty(TM)$ and $l = 1, \dots, r$.

Proposition 6.1. *Let $B : C_r^\infty(TM) \to C_s^\infty(TM)$ be a tensor field of type (r, s) and $p \in M$. Let $X_1, \dots, X_r$ and $Y_1, \dots, Y_r$ be smooth vector fields on M such that $(X_k)_p = (Y_k)_p$ for each $k = 1, \dots, r$. Then*

$$B(X_1, \dots, X_r)(p) = B(Y_1, \dots, Y_r)(p).$$

Proof. It is sufficient to prove the statement for $r = 1$ since the rest follows by induction. Put $X = X_1$ and $Y = Y_1$. Let (U, x) be a local coordinate on M. Choose a function $f \in C_0^\infty(TM)$ such that $f(p) = 1$ and support(f) is contained in U. Then define $v_1, \dots, v_m \in C^\infty(TM)$ by

$$(v_k)_q = \begin{cases} f(q) \cdot \frac{\partial}{\partial x_k}|_q & \text{if } q \in U \\ 0 & \text{if } q \notin U \end{cases}$$

Then there exist functions $\rho_k, \sigma_k \in C_0^\infty(TM)$ such that

$$f \cdot X = \sum_{k=1}^{m} \rho_k v_k \quad \text{and} \quad f \cdot Y = \sum_{k=1}^{m} \sigma_k v_k.$$

Now

$$B(X)(p) = f(p)B(X)(p) = B(f \cdot X)(p) = \sum_{k=1}^{m} \rho_k(p) B(v_k)(p)$$

and similarily $B(Y)(p) = \sum_{k=1}^m \sigma_k(p)B(v_k)(p)$. Now $X_p = Y_p$ so $\rho_k(p) = \sigma_k(p)$ for all k. This implies that $B(X)(p) = B(Y)(p)$. $\square$

By B_p we denote the restriction $B_p = B|_{\otimes_{l=1}^r T_pM}$ of B to the r-fold tensor product of T_pM given by

$$B_p : ((X_1)_p, \dots, (X_r)_p) \mapsto B(X_1, \dots, X_r)(p).$$

The tensor field B is said to be **smooth** if for all $X_1, \dots, X_r \in C^\infty(TM)$ the map $B(X_1, \dots, X_r) : M \to C_s^\infty(TM)$ with

$$B(X_1, \dots, X_r) : p \mapsto B_p((X_1)_p, \dots, (X_r)_p)$$

is smooth.

Definition 6.2. Let M be a smooth manifold. A **Riemannian metric** on M is a smooth tensor field $g : C_2^\infty(TM) \to C_0^\infty(TM)$ such that for each $p \in M$ the restriction $g_p = g|_{T_pM \otimes T_pM} : T_pM \otimes T_pM \to \mathbb{R}$ with

$$g_p : (X_p, Y_p) \mapsto g(X, Y)(p)$$

is an inner product on T_pM. The pair (M, g) is called a **Riemannian manifold**. The study of Riemannian manifolds is called **Riemannian Geometry**. Geometric properties of (M, g) which only depend on the metric g are called **intrinsic** (or **metric**) properties.

Definition 6.3. Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ be a curve in M. Then the length $L(\gamma)$ of γ is defined by

$$L(\gamma) = \int_I \sqrt{g(\gamma'(t), \gamma'(t))} dt.$$

Example 6.4. By the m-dimensional **Euclidean space** we mean the Riemannian manifold $E^m = (\mathbb{R}^m, \langle , \rangle_{\mathbb{R}^m})$ where

$$\langle u, v \rangle_{\mathbb{R}^m} = \sum_{k=1}^m u_k v_k.$$

Example 6.5. By the **punctured round sphere** we mean the Riemannian manifold

$$\Sigma^m = (\mathbb{R}^m, \frac{4}{(1 + |x|_{\mathbb{R}^m}^2)^2} \langle , \rangle_{\mathbb{R}^m}).$$

Let $\gamma : \mathbb{R}^+ \to \Sigma^m$ be the curve with $\gamma : t \mapsto (t, 0, \dots, 0)$. Then

$$L(\gamma) = 2 \int_0^\infty \frac{\sqrt{\langle \gamma', \gamma' \rangle}}{1 + |\gamma|^2} dt = 2 \int_0^\infty \frac{dt}{1 + t^2} = 2[\arctan(t)]_0^\infty = \pi$$

Example 6.6. By the **hyperbolic space** we mean the Riemannian manifold

$$H^m = (B_1^m(0), \frac{4}{(1-|x|^2_{\mathbb{R}^m})^2}\langle , \rangle_{\mathbb{R}^m})$$

where $B_1^m(0)$ is the m-dimensional open unit ball

$$B_1^m(0) = \{x \in \mathbb{R}^m \mid |x|_{\mathbb{R}^m} < 1\}.$$

Let $\gamma : (0,1) \to H^m$ be a curve given by $\gamma : t \mapsto (t, 0, \dots, 0)$. Then

$$L(\gamma) = 2\int_0^1 \frac{\sqrt{\langle \gamma', \gamma' \rangle}}{1-|\gamma|^2} dt = 2\int_0^1 \frac{dt}{1-t^2} = [\log(\frac{1+t}{1-t})]_0^1 = \infty$$

Definition 6.7. Let (M^m, g) be a Riemannian manifold and $\tilde{M}^{\tilde{m}}$ be an $\tilde{m}$-dimensional submanifold of M. Then the smooth tensor field $h : C_2^\infty(T\tilde{M}) \to C_0^\infty(T\tilde{M})$ with

$$h(X, Y) : p \mapsto g_p(X_p, Y_p).$$

is a Riemannian metric on $\tilde{M}$ called the **induced metric** on $\tilde{M}$ in (M, g).

Example 6.8. The Euclidean metric $\langle , \rangle_{\mathbb{R}^m}$ on $\mathbb{R}^m$ induces Riemannian metrics on the following submanifolds.

i. the $(m-1)$-dimensional sphere $S^{m-1} \subset \mathbb{R}^m$,
ii. the tangent bundle $TS^n \subset \mathbb{R}^m$ where $m = 2n + 2$,
iii. the n-dimensional torus $T^n \subset \mathbb{R}^{2n}$,
iv. the n-dimensional real projective space $\mathbb{R}P^n \subset \text{Sym}(\mathbb{R}^{n+1}) \subset \mathbb{R}^m$ where $m = (n+2)(n+1)/2$.

Example 6.9. On $\mathbb{C}^{n\times n}$ we have the Euclidean metric given by

$$\langle A, B \rangle = \text{Re}\{\text{trace}(\bar{A}^t \cdot B)\}.$$

This induces metrics on submanifolds of $\mathbb{C}^{n\times n}$ such as $\mathbb{R}^{n\times n}$ and all the matrix Lie groups $\mathbf{GL}(\mathbb{C}^n)$, $\mathbf{SL}(\mathbb{C}^n)$, $\mathbf{U}(n)$, $\mathbf{SU}(n)$, $\mathbf{GL}(\mathbb{R}^n)$, $\mathbf{SL}(\mathbb{R}^n)$, $\mathbf{O}(n)$ and $\mathbf{SO}(n)$.

We now need the following fact which should be known to any graduate student from a course on topology.

Fact 6.10. *Every locally compact Hausdorff space with countable basis is paracompact.*

Corollary 6.11. *Let $(M, \hat{\mathcal{A}})$ be a topological manifold. Let the collection $(U_\alpha)_{\alpha\in I}$ be an open covering of M such that for each $\alpha \in I$ the pair (U_α, ψ_α) is a chart on M. Then there exists*

i. *a locally finite open refinement $(W_\beta)_{\beta\in J}$ such that for all $\beta \in J$, W_β is an open neighbourhood for a chart $(W_\beta, \psi_\beta) \in \hat{\mathcal{A}}$, and*

ii. *a partition of unity* $(f_\beta)_{\beta\in J}$ *such that support*$(f_\beta) \subset W_\beta$.

Theorem 6.12. *Let* $(M^m, \hat{\mathcal{A}})$ *be a smooth manifold. Then there exists a Riemannian metric* g *on* M.

PROOF. For each $p \in M$ let (U_p, ϕ_p) be a chart such that $p \in U_p$. Then $(U_p)_{p\in M}$ is an open covering as in Corollary 6.11. Let $(W_\beta)_{\beta\in J}$ be a locally finite open refinement, (W_β, x_β) be charts on M and $(f_\beta)_{\beta\in J}$ be a partition of unity such that support(f_β) is contained in W_β. Let $\langle,\rangle_{\mathbb{R}^m}$ be the Euclidean metric on $\mathbb{R}^m$. Then for $\beta \in J$ define $g_\beta : C_2^\infty(TM) \to C_0^\infty(TM)$ by

$$g_\beta(\frac{\partial}{\partial x_k^\beta}, \frac{\partial}{\partial x_l^\beta})(p) = \begin{cases} f_\beta(p)\cdot\langle e_k, e_l\rangle_{\mathbb{R}^m} & \text{if } p \in W_\beta \\ 0 & \text{if } p \notin W_\beta \end{cases}$$

Then $g : C_2^\infty(TM) \to C_0^\infty(TM)$ given by $g = \sum_{\beta\in J} g_\beta$ is a Riemannian metric on M. □

Definition 6.13. Let (M, g) and (N, h) be Riemannian manifolds. A map $\phi : (M, g) \to (N, h)$ is said to be **conformal** if there exists a function $\lambda : M \to \mathbb{R}$ such that

$$e^{\lambda(p)} g_p(X_p, Y_p) = h_{\phi(p)}(d\phi_p(X_p), d\phi_p(Y_p)),$$

for all $X, Y \in C^\infty(TM)$ and $p \in M$. The function e^λ is called the **conformal factor** of ϕ. A conformal map with $\lambda \equiv 0$ is said to be **isometric**. An isometric diffeomeorphism is called an **isometry**.

Example 6.14. Equip the orthogonal group $\mathbf{O}(m) \subset \mathbb{R}^{m\times m}$ with the induced metric given by $\langle A, B\rangle := \text{trace}(A^t \cdot B)$. For $x \in \mathbf{O}(m)$ the left translation $L_x : \mathbf{O}(m) \to \mathbf{O}(m)$ by x is given by $L_x : y \mapsto xy$. The tangent space $T_y\mathbf{O}(m)$ of $\mathbf{O}(m)$ at y is $T_y\mathbf{O}(m) = \{y \cdot Z |\ Z + Z^t = 0\}$ and the differential $(dL_x)_y : T_y\mathbf{O}(m) \to T_{xy}\mathbf{O}(m)$ is given by $(dL_x)_y : yZ \mapsto xyZ$. We then have

$$\begin{aligned} \langle (dL_x)_y(yZ), (dL_x)_y(yW)\rangle_{xy} &= \text{trace}((xyZ)^t xyW) \\ &= \text{trace}(Z^t y^t x^t xyW) \\ &= \text{trace}(yZ)^t(yW). \\ &= \langle yZ, yW\rangle_y \end{aligned}$$

This shows that for each $x \in \mathbf{O}(m)$ the left translation $L_x : \mathbf{O}(m) \to \mathbf{O}(m)$ is an isometry.

Definition 6.15. Let G be a Lie group. A Riemannian metric $\langle,\rangle$ on G is said to be left-invariant if for each $x \in G$ the left-translation $L_x : G \to G$ is an isometry.

Proposition 6.16. *Let G be a Lie group and $\langle,\rangle_e$ be an inner product on the tangent space T_eG at the neutral element e. Then for each $x \in G$ the bilinear map $\langle,\rangle_x : T_xG \times T_xG \to \mathbb{R}$ with*

$$\langle (dL_x)_e(Z_e), (dL_x)_e(W_e)\rangle_x = \langle Z_e, W_e\rangle_e$$

is an inner product on the tangent space T_xG. The smooth tensor field $\langle,\rangle : C_2^\infty(TG) \to C_0^\infty(TG)$ given by

$$\langle,\rangle : (Z, W) \mapsto (\langle Z, W\rangle : x \mapsto \langle Z_x, W_x\rangle_x)$$

is a left-invariant Riemannian metric on G.

PROOF. See Exercise 6.3. □

Example 6.17. Let $(S^m, \langle,\rangle_{\mathbb{R}^{m+1}})$ be the standard sphere. Let the linear space of symmetric $\mathbb{R}^{(m+1)\times(m+1)}$ matrices $\mathrm{Sym}(\mathbb{R}^{m+1})$ be equipped with the metric g given by

$$g(A, B) = \frac{1}{8} \cdot \mathrm{trace}(A^t \cdot B).$$

As in Example 5.3 define a map $\phi : S^m \to \mathrm{Sym}(\mathbb{R}^{m+1})$ by

$$\phi : p \mapsto (\rho_p : q \mapsto 2\langle q, p\rangle p - q).$$

Let $\alpha, \beta : \mathbb{R} \to S^m$be two curves such that $\alpha(0) = p = \beta(0)$ and put $a = \alpha'(0)$, $b = \beta'(0)$. Then for $\gamma \in \{\alpha, \beta\}$ we have

$$d\phi_p(\gamma'(0)) = (q \mapsto 2\langle q, \gamma'(0)\rangle p + 2\langle q, p\rangle \gamma'(0)).$$

Let $\mathcal{B}$ be an orthonormal basis for $\mathbb{R}^{m+1}$, then

$$\begin{aligned} g(d\phi_p(a), d\phi_p(b)) &= \frac{1}{8}\,\mathrm{trace}(d\phi_p(a)^t \cdot d\phi_p(b)) \\ &= \frac{1}{2}\sum_{q\in\mathcal{B}} \langle \langle q, a\rangle p + \langle q, p\rangle a, \langle q, b\rangle p + \langle q, p\rangle b\rangle \\ &= \frac{1}{2}\sum_{q\in\mathcal{B}} \{\langle p, p\rangle\langle a, q\rangle\langle q, b\rangle + \langle a, b\rangle\langle p, q\rangle\langle p, q\rangle\} \\ &= \frac{1}{2}\{\langle a, b\rangle + \langle a, b\rangle\} \\ &= \langle a, b\rangle \end{aligned}$$

This proves that the immersion ϕ is isometric.

The following result was proved by J. Nash in his famous paper: *The embedding problem for Riemannian manifolds*, Ann. of Math. **63** (1956), 20-63. It implies that every Riemannian manifold can be realized as a submanifold of a Euclidean space.

Deep Result 6.18. *For $3 \leq r \leq \infty$ let (M, g) be a Riemannian C^r-manifold. Then there exists an isometric C^r-embedding into a Euclidean space.*

We will now see that parametrizations can be very useful tools for the study of the intrinsic geometry of a Riemannian manifold (M, g_M). Let $p \in M$, $\hat{\psi} : U \to M$ be a parametrization of M with $q \in U$ and $\hat{\psi}(q) = p$. The differential $d\hat{\psi}_q : T_q\mathbb{R}^m \to T_pM$ is bijective so there exist neighbourhoods U_q of q and U_p of p such that the restriction $\psi = \hat{\psi}|_{U_q} : U_q \to U_p$ is a diffeomorphism. On U_q we have the frame $\{e_1, \dots, e_m\}$ for TU_q so $\{d\hat{\psi}(e_1), \dots, d\hat{\psi}(e_m)\}$ is a local frame for TM over U_p. We then define the pull-back metric $g = \hat{\psi}^* g_M$ on U_q by

$$g_{kl} = g(e_k, e_l) = g_M(d\hat{\psi}(e_k), d\hat{\psi}(e_l)).$$

Then $\hat{\psi} : U_q \to U_p$ is an isometry so the intrinsic geometry of (U_q, g) and that of (U_p, g_M) are exactly the same.

Example 6.19. Let G be a matrix Lie group and e be the neutral element of G. Let $\{X_1, \dots, X_m\}$ be a basis for the Lie algebra T_eG. For $x \in G$ define $\psi_x : \mathbb{R}^m \to G$ by

$$\psi_x : (t_1, \dots, t_m) \mapsto L_x(\prod_{k=1}^{m} \exp(t_k X_k)).$$

Then $(d\psi_x)_0(e_k) = X_k(x)$ for all k. This means that the differential $(d\psi_x)_0 : T_0\mathbb{R}^m \to T_xG$ is an isomorphism so there exist open neighbourhoods U_0 of 0 and U_x of x such that the restriction of ψ to U_0 is bijective onto its image U_x.

Definition 6.20. Let (M, g) be a Riemannian manifold and $\tilde{M}$ be a submanifold of M. For a point $p \in \tilde{M}$ we define the **normal space** $N_p\tilde{M}$ of $\tilde{M}$ at p by

$$N_p\tilde{M} = \{v \in T_pM |\ g_p(v, w) = 0 \ \text{ for all } \ w \in T_p\tilde{M}\}.$$

For all p we have the orthogonal decomposition

$$T_pM = T_p\tilde{M} \oplus N_p\tilde{M}.$$

The **normal bundle** of $\tilde{M}$ in M is defined by

$$N\tilde{M} = \{(p, v) |\ p \in \tilde{M}, \quad v \in N_p\tilde{M}\}.$$

Theorem 6.21. *Let (M^m, g) be a Riemannian manifold and $\tilde{M}^{\tilde{m}}$ be a smooth submanifold of M. Then the normal bundle $(N\tilde{M}, \tilde{M}, \pi)$ is a smooth $(m - \tilde{m})$-dimensional vector bundle over $\tilde{M}$.*

PROOF. See Exercise 6.5. □

Example 6.22. The orthogonal group $\mathbf{O}(m)$ is a subset of the linear space $\mathbb{R}^{m\times m}$ equipped with the Riemannian metric

$$\langle A, B\rangle_{\mathbb{R}^{m\times m}} = \text{trace}(A^t B).$$

We have already seen that the tangent space $T_e\mathbf{O}(m)$ of $\mathbf{O}(m)$ at the neutral element e is

$$T_e\mathbf{O}(m) = \{Z \in \mathbb{R}^{m\times m} | Z + Z^t = 0\}$$

and the tangent bundle $T\mathbf{O}(m)$ of $\mathbf{O}(m)$ is given by

$$T\mathbf{O}(m) = \{(x, xZ)|\ x \in \mathbf{O}(m), \quad Z \in T_e\mathbf{O}(m)\}.$$

The space $\mathbb{R}^{m\times m}$ has a linear decomposition

$$\mathbb{R}^{m\times m} = \text{Sym}(\mathbb{R}^m) \oplus T_e\mathbf{O}(m)$$

and every element $X \in \mathbb{R}^{m\times m}$ can be decomposed $X = X^N + X^T$ in its symmetric and skew-symmetric parts given by

$$X^N = \frac{1}{2}(X + X^t) \ \text{ and } \ X^T = \frac{1}{2}(X - X^t).$$

If $Z \in T_e\mathbf{O}(m)$ and $W \in \text{Sym}(\mathbb{R}^m)$ then

$$\begin{aligned} \langle Z, W\rangle_{\mathbb{R}^{m\times m}} &= \text{trace}(Z^t W) = \text{trace}(W^t Z) \\ &= \text{trace}(Z W^t) = \text{trace}(-Z^t W) \\ &= -\langle Z, W\rangle_{\mathbb{R}^{m\times m}} \end{aligned}$$

This implies that the normal bundle $N\mathbf{O}(m)$ of $\mathbf{O}(m)$ in $\mathbb{R}^{m\times m}$ is given by

$$N\mathbf{O}(m) = \{(x, xW)|\ x \in \mathbf{O}(m), \quad W \in \text{Sym}(\mathbb{R}^m)\}.$$

Exercises

Exercise 6.1. For $m \in \mathbb{N}^+$ let the **stereographic projection**

$$\pi_m : (S^m - \{(1, 0, \dots, 0)\}, \langle , \rangle_{\mathbb{R}^{m+1}}) \to (\mathbb{R}^m, \frac{4}{(1 + |x|^2)^2} \langle , \rangle_{\mathbb{R}^m})$$

be given by

$$\pi_m : (x_0, \dots, x_m) \mapsto \frac{1}{1 - x_0}(x_1, \dots, x_m).$$

Prove that π_m is an isometry for each m.

Exercise 6.2. Let $B_1^2(0)$ be the open unit disk in the complex plane equipped with the hyperbolic metric $g(,) = 4/(1 - |z|^2)^2 \langle , \rangle_{\mathbb{R}^2}$. Prove that the map

$$\pi : B_1^2(0) \to (\{z \in \mathbb{C} | \ \mathrm{Im}(z) > 0\}, \frac{1}{\mathrm{Im}(z)^2} \langle , \rangle_{\mathbb{R}^2})$$

given by $\pi : z \mapsto (z + i)/(iz + 1)$ is an isometry.

Exercise 6.3. Find a proof for Proposition 6.16.

Exercise 6.4. On the real general linear group $\mathbf{GL}(\mathbb{R}^m)$ we define metrics g, h by

$$g_x(xZ, xW) = \mathrm{trace}((xZ)^t \cdot xW) \quad \text{and} \quad h_x(xZ, xW) = \mathrm{trace}(Z^t \cdot W).$$

They induce metrics $\hat{g}, \hat{h}$ on the real special linear group $\mathbf{SL}(\mathbb{R}^m)$.

i. Which of the metrics $g, h, \hat{g}, \hat{h}$ are left-invariant?
ii. Find the normal space $N_e\mathbf{SL}(\mathbb{R}^m)$ of $\mathbf{SL}(\mathbb{R}^m)$ in $\mathbf{GL}(\mathbb{R}^m)$ w.r.t g
iii. Find the normal bundle $N\mathbf{SL}(\mathbb{R}^m)$ of $\mathbf{SL}(\mathbb{R}^m)$ in $\mathbf{GL}(\mathbb{R}^m)$ w.r.t h.

Exercise 6.5. Find a proof for Theorem 6.21.

The Levi-Civita Connection

Definition 7.1. Let (E, M, π) be a smooth vector bundle over M. A **connection** on (E, M, π) is a map $\hat{\nabla} : C^\infty(TM) \times C^\infty(E) \to C^\infty(E)$ such that

i. $\hat{\nabla}_X(\lambda \cdot v + \mu \cdot w) = \lambda \cdot \hat{\nabla}_X v + \mu \cdot \hat{\nabla}_X w$,
ii. $\hat{\nabla}_X(f \cdot v) = X(f) \cdot v + f \cdot \hat{\nabla}_X v$,
iii. $\hat{\nabla}_{(f \cdot X + g \cdot Y)} v = f \cdot \hat{\nabla}_X v + g \cdot \hat{\nabla}_Y v$.

for all $\lambda, \mu \in \mathbb{R}$, $X, Y \in C^\infty(TM)$, $v, w \in C^\infty(E)$ and $f, g \in C_0^\infty(TM)$, and the map $(\hat{\nabla}_X v) : M \to E$ with $(\hat{\nabla}_X v) : p \mapsto (\hat{\nabla}_X v)_p$ is smooth. A section $v \in C^\infty(E)$ is said to be **parallel** with respect to the connection $\hat{\nabla}$ if $\hat{\nabla}_X v = 0$ for all $X \in C^\infty(TM)$.

Definition 7.2. Let M be a smooth manifold and $\hat{\nabla}$ be a connection on the tangent bundle (TM, M, π). Then we define the **torsion** $T : C_2^\infty(TM) \to C_1^\infty(TM)$ of $\hat{\nabla}$ by

$$T(X, Y) = \hat{\nabla}_X Y - \hat{\nabla}_Y X - [X, Y],$$

where $[,]$ is the Lie bracket on $C^\infty(TM)$.

Definition 7.3. Let M be a smooth manifold. A connection $\hat{\nabla}$ on the tangent bundle (TM, M, π) is said to be **torsion-free** if the corresponding torsion T vanishes i.e. $T(X, Y) = 0$ for all $X, Y \in C^\infty(TM)$. If g is a Riemannian metric on M, then $\hat{\nabla}$ is said to be **metric** (or **compatible with** g) if

$$X(g(Y, Z)) = g(\hat{\nabla}_X Y, Z) + g(Y, \hat{\nabla}_X Z)$$

for all $X, Y, Z \in C^\infty(TM)$.

Lemma 7.4. *Let M be a smooth manifold and $[,]$ be the Lie bracket on the tangent bundle TM. Then*

i. $[X, f \cdot Y] = X(f) \cdot Y + f \cdot [X, Y]$,
ii. $[f \cdot X, Y] = -Y(f) \cdot X + f \cdot [X, Y]$

for all $X, Y \in C^\infty(TM)$ and $f \in C_0^\infty(TM)$, PROOF. If $h \in C_0^\infty(TM)$, then

$$\begin{aligned}[X, f \cdot Y](h) &= X(f \cdot Y(h)) - f \cdot Y(X(h)) \\ &= X(f) \cdot Y(h) + f \cdot X(Y(h)) - f \cdot Y(X(h)) \\ &= (X(f) \cdot Y + f \cdot [X, Y])(h)\end{aligned}$$

This proves the first statement and the second follows from the skew-symmetry of the Lie bracket. □

Theorem 7.5. *Let (M, g) be a Riemannian manifold and let the map $\nabla : C_2^\infty(TM) \to C^\infty(TM)$ be given by*

$$\begin{aligned}g(\nabla_X Y, Z) &= \frac{1}{2}\{X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\quad +g(Z, [X, Y]) + g(Y, [Z, X]) - g(X, [Y, Z])\}.\end{aligned}$$

Then ∇ is a connection on the tangent bundle (TM, M, π).

PROOF. It follows from Definition 3.1, Theorem 4.15 and the fact that g is a tensor field that

$$g(\nabla_X(\lambda \cdot Y_1 + \mu \cdot Y_2), Z) = \lambda \cdot g(\nabla_X Y_1, Z) + \mu \cdot g(\nabla_X Y_2, Z)$$

and

$$g(\nabla_{Y_1 + Y_2} X, Z) = g(\nabla_{Y_1} X, Z) + g(\nabla_{Y_2} X, Z)$$

for all $\lambda, \mu \in \mathbb{R}$ and $X, Y_1, Y_2, Z \in C^\infty(TM)$. Furthermore we have

$$\begin{aligned}&g(\nabla_X fY, Z) \\ = &\frac{1}{2}\{X(f \cdot g(Y, Z)) + f \cdot Y(g(Z, X)) - Z(f \cdot g(X, Y)) \\ &+g(Z[X, f \cdot Y]) + f \cdot g(Y, [Z, X]) - g(X, [f \cdot Y, Z])\} \\ = &\frac{1}{2}\{X(f) \cdot g(Y, Z) + f \cdot X(g(Y, Z)) + f \cdot Y(g(Z, X)) \\ &-Z(f) \cdot g(X, Y) - f \cdot Z(g(X, Y)) + g(Z, X(f) \cdot Y + f \cdot [X, Y]) \\ &+f \cdot g(Y, [Z, X]) - g(X, -Z(f) \cdot Y + f \cdot [Y, Z])\} \\ = &X(f) \cdot g(Y, Z) + f \cdot g(\nabla_X Y, Z) \\ = &g(X(f) \cdot Y + f \cdot \nabla_X Y, Z)\end{aligned}$$

and

$$
\begin{aligned}
& g(\nabla_{f \cdot X} Y, Z) \\
= \ & \frac{1}{2}\{f \cdot X(g(Y,Z)) + Y(f \cdot g(Z,X)) - Z(f \cdot g(X,Y)) \\
& + g(Z,[f \cdot X, Y]) + g(Y,[Z, f \cdot X]) - f \cdot g(X,[Y,Z])\} \\
= \ & \frac{1}{2}\{f \cdot X(g(Y,Z)) + Y(f) \cdot g(Z,X) + f \cdot Y(g(Z,X)) \\
& - Z(f) \cdot g(X,Y) - f \cdot Z(g(X,Y)) \\
& + g(Z, -Y(f) \cdot X) + g(Z, f \cdot [X,Y]) + g(Y, Z(f) \cdot X) \\
& f \cdot g(Y,[Z,X]) - f \cdot g(X,[Y,Z])\} \\
= \ & f \cdot g(\nabla_X Y, Z).
\end{aligned}
$$

This proves that ∇ is a connection on the tangent bundle (TM, M, π).

□

Definition 7.6. The connection $\nabla : C^\infty(TM) \times C^\infty(TM) \to C_1^\infty(TM)$ defined in Theorem 7.5 is called the **Levi-Civita connection**.

The next result is called **The Fundamental Theorem of Riemannian Geometry**

Theorem 7.7. *Let (M, g) be a Riemannian manifold. Then the Levi-Civita connection is a unique torsion-free and metric connection on (TM, M, π).*

Proof. The difference $g(\nabla_X Y, Z) - g(\nabla_Y X, Z)$ equals twice the skew-symmetric (w.r.t the pair (X, Y)) part of the right hand side in Theorem 7.5. This is the same as

$$= \frac{1}{2}\{g(Z,[X,Y]) - g(Z,[Y,X])\} = g(Z,[X,Y]).$$

This proves that the Levi-Civita connection is torsion-free.

The sum $g(\nabla_X Y, Z) + g(\nabla_X Z, Y)$ equals twice the symmetric (w.r.t the pair (Y, Z)) part on the right hand side of Theorem 7.5. This is exactly

$$= \frac{1}{2}\{X(g(Y,Z)) + X(g(Z,Y))\} = X(g(Y,Z)).$$

This shows that the Levi-Civita connection is compatible with the Riemannian metric g on M.

Let us now assume that $\hat{\nabla}$ is a torsion-free and metric connection. Then it is easily seen that the following equations hold

$$g(\hat{\nabla}_X Y, Z) = X(g(Y,Z)) - g(Y, \hat{\nabla}_X Z),$$

$$\begin{aligned} g(\hat\nabla_X Y, Z) &= g([X,Y],Z) + g(\hat\nabla_Y X, Z) \\ &= g([X,Y],Z) + Y(g(X,Z)) - g(X, \hat\nabla_Y Z), \end{aligned}$$

$$\begin{aligned} 0 &= -Z(g(X,Y)) + g(\hat\nabla_Z X, Y) + g(X, \hat\nabla_Z Y) \\ &= -Z(g(X,Y)) + g(\hat\nabla_X Z + [Z,X], Y) + g(X, \hat\nabla_Y Z - [Y,Z]). \end{aligned}$$

We add these equations and obtain

$$\begin{aligned} 2\cdot g(\hat\nabla_X Y, Z) &= \{X(g(Y,Z) + g([X,Y],Z) + Y(g(X,Z)) \\ &\quad -Z(g(X,Y)) + g(([Z,X],Y) - g(X,[Y,Z])\} \\ &= 2\cdot g(\nabla_X Y, Z). \end{aligned}$$

This implies that $\hat\nabla = \nabla$ and thereby proves the uniqueness of ∇. □

A connection $\hat\nabla$ on (TM, M, π) can be thought of as a rule for differentiating a vector field $Y \in C^\infty(TM)$ in the direction of another $X \in C^\infty(TM)$ by $\hat\nabla_X Y$. The last theorem shows that given a Riemannian metric g on M there is only one way of doing this in a **metric** and **torsion-free** manner. The Levi-Civita connection ∇ is by definition determined by the metric g so it is an intrinsic object.

Definition 7.8. Let G be a Lie group. For a left-invariant vector field $X \in \mathfrak{g}$ we define a map $\mathrm{ad}(X) : \mathfrak{g} \to \mathfrak{g}$ by

$$\mathrm{ad}(X) : Z \mapsto [X, Z].$$

Proposition 7.9. *Let $(G, \langle,\rangle)$ be a Lie group equipped with a left-invariant metric such that for all $X \in \mathfrak{g}$, $ad(X)$ is skew-symmetric with respect to $\langle,\rangle$ i.e.*

$$\langle ad(X)Y, Z\rangle = -\langle Y, ad(X)Z\rangle$$

for all $X, Y, Z \in \mathfrak{g}$. Then the Levi-Civita connection of $(G, \langle,\rangle)$ is given by $\nabla_X Y = \frac{1}{2}[X,Y]$ for all left-invariant $X, Y \in \mathfrak{g}$.

Proof. If $X, Y, Z \in \mathfrak{g}$ then it follows from the fact that $\langle,\rangle$ is left-invariant that the function $\langle Y, Z\rangle : G \to \mathbb{R}$ is constant so $X(\langle Y, Z\rangle) = 0$. It then follows from the definition of the Levi-Civita connection and the fact that ad is skew-symmetric that

$$\begin{aligned} \langle \nabla_X Y - \frac{1}{2}[X,Y], Z\rangle &= \frac{1}{2}\{\langle Y, [Z,X]\rangle - \langle X, [Y,Z]\rangle\} \\ &= \frac{1}{2}\{\langle Y, \mathrm{ad}(Z)X\rangle + \langle \mathrm{ad}(Z)Y, X\rangle\} \\ &= 0 \end{aligned}$$

□

Example 7.10. Let (M, g_M, ∇) be a Riemannian manifold with Levi-Civita connection. Further let (U, x) be a local coordinate on M and define $X_i = \frac{\partial}{\partial x_i} \in C^\infty(TU)$. Then $\{X_1, \dots, X_m\}$ is a local frame of TM on U. For (U, x) we define the **Christoffel symbols** $\Gamma_{ij}^k : U \to \mathbb{R}$ of ∇ with respect to (U, x) by

$$\sum_{k=1}^{m} \Gamma_{ij}^k X_k = \nabla_{X_i} X_j.$$

On the subset $x(U)$ of $\mathbb{R}^m$ we define the metric g by

$$g_{ij} = g(e_i, e_j) = g_M\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right).$$

Following Lemma 4.17 and Exercise 4.5 we obtain for the differential dx of x

$$dx([X_i, X_j]) = [dx(X_i), dx(X_j)] = [e_i, e_j] = 0$$

so $[X_i, X_j] = 0$ since dx is bijective. From the definition of the Levi-Civita connection we get

$$\begin{aligned}
\sum_{k=1}^{m} \Gamma_{ij}^k g_{kl} &= \langle \sum_{k=1}^{m} \Gamma_{ij}^k X_k, X_l \rangle \\
&= \langle \nabla_{X_i} X_j, X_l \rangle \\
&= \frac{1}{2}\{X_i \langle X_j, X_l \rangle + X_j \langle X_l, X_i \rangle - X_l \langle X_i, X_j \rangle\} \\
&= \frac{1}{2}\left\{\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\right\}.
\end{aligned}$$

If $g^{kl} = (g^{-1})_{kl}$ then

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{m} g^{kl} \left\{\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\right\}.$$

Definition 7.11. Let M be a smooth manifold, $\tilde{M}$ be a submanifold and $\tilde{X} \in C^\infty(T\tilde{M})$ be a vector field on $\tilde{M}$. Let U be an open subset of M such that $U \cap \tilde{M} \neq \emptyset$. A **local extension** of $\tilde{X}$ on U is a vector field $X \in C^\infty(TU)$ such that $\tilde{X}_p = X_p$ for all $p \in \tilde{M}$. If $U = M$ then X is called a **global extension**.

Fact 7.12. *Every vector field $\tilde{X} \in C^\infty(T\tilde{M})$ has a global extension $X \in C^\infty(TM)$.*

Remark 7.13. Let (M, g) be a Riemannian manifold and $\tilde{M}$ be a submanifold equipped with the induced metric $\tilde{g}$. Let $Z \in C^\infty(TM)$ be a vector field on M and $\tilde{Z} = Z|_{\tilde{M}} : \tilde{M} \to TM$ be the restriction of Z to $\tilde{M}$. Note that $\tilde{Z}$ is not necessarily an element of $C^\infty(T\tilde{M})$. For each $p \in \tilde{M}$ the tangent vector $\tilde{Z}_p \in T_pM$ can be decomposed $\tilde{Z}_p = (\tilde{Z}_p)^T + (\tilde{Z}_p)^N$ in a unique way such that $(\tilde{Z}_p)^T \in T_p\tilde{M}$ and $(\tilde{Z}_p)^N \in N_p\tilde{M}$. For the vector field we write $\tilde{Z} = \tilde{Z}^T + \tilde{Z}^N$.

Let $\tilde{X}, \tilde{Y} \in C^\infty(T\tilde{M})$ be vector fields on $\tilde{M}$ and $X, Y \in C^\infty(TM)$ be their extensions onto M. If $p \in \tilde{M}$ then $(\nabla_X Y)_p$ only depends on the value $X_p = \tilde{X}_p$ and the value of Y along some curve $\gamma : (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = X_p = \tilde{X}_p$. For this see Remark 8.3. Hence we can choose the curve γ such that $\gamma((-\epsilon, \epsilon))$ is contained in $\tilde{M}$. Then $\tilde{Y}_{\gamma(t)} = Y_{\gamma(t)}$ for $t \in (-\epsilon, \epsilon)$. This implies that $(\nabla_X Y)_p$ only depends on $\tilde{X}_p$ and the value of $\tilde{Y}$ along γ, but not on the way $\tilde{X}$ and $\tilde{Y}$ are extended. This implies that the following maps are well-defined.

Definition 7.14. For the above situation we define

$$\tilde{\nabla} : C_2^\infty(T\tilde{M}) \to C_1^\infty(T\tilde{M}) \quad \text{and} \quad B : C_2^\infty(T\tilde{M}) \to C_1^\infty(N\tilde{M})$$

with

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = (\nabla_X Y)^T \quad \text{and} \quad B(\tilde{X}, \tilde{Y}) = (\nabla_X Y)^N.$$

It is easily proved that B is symmetric and hence tensorial in both its arguments, see Exercise 7.4. B is called the **second fundamental form** of $\tilde{M}$ in (M, g).

The Levi-Civita connection on (M, g) induces the Levi-Civita connection on any submanifold $\tilde{M}$ and a metric connection on its normal bundle $N\tilde{M}$:

Theorem 7.15. *Let (M, g) be a Riemannian manifold and $\tilde{M}$ be a submanifold of M with the induced metric $\tilde{g}$. Then $\tilde{\nabla} : C_2^\infty(T\tilde{M}) \to C_1^\infty(T\tilde{M})$ is the Levi-Civita connection of the submanifold $(\tilde{M}, \tilde{g})$.*

PROOF. See Exercise 7.5. □

Proposition 7.16. *Let (M, g) be a Riemannian manifold and $\tilde{M}$ be a submanifold with the induced metric $\tilde{g}$. Let $X, Z \in C^\infty(TM)$ be vector fields extending $\tilde{X} \in C^\infty(T\tilde{M})$ and $\tilde{Z} \in C^\infty(N\tilde{M})$. Then the map $\bar{\nabla} : C^\infty(T\tilde{M}) \times C^\infty(N\tilde{M}) \to C^\infty(N\tilde{M})$ given by*

$$\bar{\nabla}_{\tilde{X}}\tilde{Z} = (\nabla_X Z)^N$$

is a well-defined connection on the normal bundle $N\tilde{M}$. Furthermore

$$\tilde{X}(\langle \tilde{Z}, \tilde{W} \rangle) = \langle \bar{\nabla}_{\tilde{X}}\tilde{Z}, \tilde{W} \rangle + \langle \tilde{Z}, \bar{\nabla}_{\tilde{X}}\tilde{W} \rangle$$

for all $\tilde{X} \in C^\infty(T\tilde{M})$ and $\tilde{Z}, \tilde{W} \in C^\infty(N\tilde{M})$.

PROOF. See Exercise 7.6. □

Exercises

Exercise 7.1. Prove that the torsion T in Definition 7.2 is a tensor field of type $(2,1)$.

Exercise 7.2. Let $\mathbf{SO}(m)$ be the special orthogonal group equipped with the metric

$$\langle X, Y \rangle = \frac{1}{2}\text{trace}(X^t \cdot Y).$$

Prove that $\langle , \rangle$ is left-invariant and that for left-invariant vector fields $X, Y \in \mathfrak{so}(m)$ we have $\nabla_X Y = \frac{1}{2}[X, Y]$. Let A, B, C be elements of the Lie algebra $\mathfrak{so}(3)$ with

$$A_e = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B_e = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, C_e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Prove that $\{A, B, C\}$ is an orthonormal basis for $\mathfrak{so}(3)$ and calculate $(\nabla_A B)_e$, $(\nabla_B C)_e$ and $(\nabla_C A)_e$.

Exercise 7.3. Let $\mathbf{SL}(\mathbb{R}^2)$ be the real special linear group equipped with the metric

$$\langle X, Y \rangle_p = \text{trace}((p^{-1}X)^t \cdot (p^{-1}Y)).$$

Find a formula for the Levi-Civita connection $\nabla_X Y$ for $X, Y \in \mathfrak{sl}(\mathbb{R}^2)$. Calculate $(\nabla_A B)_e, (\nabla_B C)_e$ and $(\nabla_C A)_e$ where $A, B, C \in \mathfrak{sl}(\mathbb{R}^2)$ are given by

$$A_e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, B_e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, C_e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Exercise 7.4. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and $(\tilde{M}, \tilde{g})$ be a submanifold with the induced metric. Prove that the second fundamental form B of $\tilde{M}$ in M is symmetric and tensorial in both its arguments.

Exercise 7.5. Find a proof for Theorem 7.15.

Exercise 7.6. Find a proof for Proposition 7.16.

Geodesics

Definition 8.1. Let (TM, M, π) be the tangent bundle of the smooth manifold M. A **vector field X along a curve** $\gamma : I \to M$ is a curve $X : I \to TM$ such that $\pi \circ X = \gamma$. We denote by $C^\infty_\gamma(TM)$ the set of all smooth vector fields along γ. It is easily seen that the operations $\cdot$ and $+$ given by

i. $(\lambda \cdot X)(t) = \lambda \cdot X(t)$,
ii. $(X + Y)(t) = X(t) + Y(t)$,

make $(C^\infty_\gamma(TM), +, \cdot)$ into a vector space.

Proposition 8.2. *Let (M, g) be a smooth Riemannian manifold and $\gamma : I \to M$ be a smooth curve. Then there exists a unique operator $\frac{D}{dt} : C^\infty_\gamma(TM) \to C^\infty_\gamma(TM)$ such that for all $\lambda, \mu \in \mathbb{R}$ and $f \in C^\infty(I, \mathbb{R})$,*

i. $\frac{D}{dt}(\lambda \cdot X + \mu \cdot Y) = \lambda \cdot (\frac{D}{dt}X) + \mu \cdot (\frac{D}{dt}Y)$,
ii. $\frac{D}{dt}(f \cdot Y) = \frac{df}{dt} \cdot Y + f \cdot (\frac{D}{dt}Y)$, *and*
iii. *if J_0 is an open subset of I such that $t_0 \in J_0$ and $X \in C^\infty(TM)$ is a vector field with $X_{\gamma(t)} = Y(t)$ for all $t \in J_0$ then*

$$(\frac{D}{dt}Y)(t_0) = (\nabla_{\dot{\gamma}} X)_{\gamma(t_0)}.$$

Proof. Let us first prove the uniqueness, so for the moment we assume that such an operator $\frac{D}{dt}$ exists. For a point $t_0 \in I$ choose a chart (U, x) on M such that $\gamma(t_0) \in U$. Then a vector field Y along γ can be written in the form

$$Y(t) = \sum_{k=1}^m \alpha_k(t) \frac{\partial}{\partial x_k}|_{\gamma(t)}$$

for some functions $\alpha_k \in C^\infty(I, \mathbb{R})$. The second condition implies that

$$(1) \qquad \frac{D}{dt}Y(t) = \sum_{k=1}^m \alpha_k(t)(\frac{D}{dt}\frac{\partial}{\partial x_k})(t) + \sum_{k=1}^m \dot{\alpha}_k(t) \frac{\partial}{\partial x_k}|_{\gamma(t)}.$$

Let $x \circ \gamma(t) = (x^1(t), \dots, x^m(t))$ then

$$\dot{\gamma}(t) = \sum_{k=1}^m (\dot{x}^k)(t) \frac{\partial}{\partial x_k}|_{\gamma(t)}$$

and the third condition for $\frac{D}{dt}$ imply that

$$(2)\qquad (\frac{D}{dt}\frac{\partial}{\partial x_j})_{\gamma(t)} = (\nabla_{\dot\gamma}\frac{\partial}{\partial x_j})_{\gamma(t)} = \sum_{k=1}^m (\dot x^k)(t)(\nabla_{\frac{\partial}{\partial x_k}}\frac{\partial}{\partial x_j})_{\gamma(t)}.$$

Together equations (1) and (2) give

$$(3)\qquad \frac{D}{dt}Y(t) = \sum_{k=1}^m \{\dot\alpha_k(t) + \sum_{i,j=1}^m \Gamma_{ij}^k \circ \gamma(t)(\dot x^i)(t)\alpha_j(t)\}\frac{\partial}{\partial x_k}|_{\gamma(t)}.$$

This shows that the operator $\frac{D}{dt}$ is uniquely determined.

It is easily seen that if we use equation (3) for defining an operator $\frac{D}{dt}$ then it satisfies the necessary conditions of Proposition 8.2. This proves the existence of the operator $\frac{D}{dt}$. □

Remark 8.3. It follows from the fact that the Levi-Civita connection is tensorial in its first argument i.e.

$$\nabla_{f\,\cdot\,X}Y = f\cdot\nabla_X Y$$

and Proposition 8.2 that the value $(\nabla_X Y)_p$ of $\nabla_X Y$ at p only depends on the value of X_p of X at p and the values of Y along some curve γ satisfying $\gamma(0)=p$ and $\dot\gamma(0)=X_p$. The equality

$$(\frac{D}{dt}Y)(t_0) = (\nabla_{\dot\gamma}X)_{\gamma(t_0)}$$

in Proposition 8.2 allows us to use the notation $\nabla_{\dot\gamma}Y$ for $\frac{D}{dt}Y$.

Definition 8.4. Let (M,g) be a Riemannian manifold and $\gamma : I \to M$ be a C^2-curve. A vector field X along γ is said to be **parallel** along γ if

$$\nabla_{\dot\gamma}X = 0.$$

The curve $\gamma : I \to M$ is called a **geodesic** if the vector field $\dot\gamma$ is parallel along γ i.e.

$$\nabla_{\dot\gamma}\dot\gamma = 0.$$

Proposition 8.5. *Let (M,g) be a Riemannian manifold, $\gamma : I \to M$ be a smooth curve and X,Y be parallel vector fields along γ. Then the function $g(X,Y) : I \to \mathbb{R}$ given by $t \mapsto g_{\gamma(t)}(X_{\gamma(t)}, Y_{\gamma(t)})$ is constant. In particular if γ is a geodesic then $g(\dot\gamma,\dot\gamma)$ is constant along γ.*

Proof. Using the fact that the Levi-Civita connection is metric we obtain

$$\frac{d}{dt}(g(X,Y)) = g(\nabla_{\dot\gamma}X, Y) + g(X, \nabla_{\dot\gamma}Y) = 0.$$

This proves that the function $g(X,Y)$ is constant along γ. □

Corollary 8.6. *Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ be a smooth curve. If $X_1, \dots, X_m$ are parallel vector fields along γ such that for some $p \in \gamma(I)$ the set $\{(X_1)_p, \dots, (X_m)_p\}$ is a (orthonormal) basis for the tangent space T_pM, then the set $\{(X_1)_q, \dots, (X_m)_q\}$ is a (orthonormal) basis for T_qM for all $q \in \gamma(I)$.*

Theorem 8.7. *Let (M, g) be a Riemannian manifold and $I = [a, b]$ be an interval on the real line $\mathbb{R}$. Further let $\gamma : I \to M$ be a smooth curve, $t_0 \in I$ and $X_0 \in T_{\gamma(t_0)}M$. Then there exists a unique parallel vector field Y along γ such that $X_0 = Y_{\gamma(t_0)}$.*

Proof. Without loss of generality we may assume that the image of γ lies in a chart (U, x). On the interval I the tangent field $\dot{\gamma}$ is represented in our local coordinate by

$$\dot{\gamma}(t) = \sum_{i=1}^{m} \rho_i(t) \frac{\partial}{\partial x_i}|_{\gamma(t)}$$

with some functions $\rho_i \in C^\infty(I, \mathbb{R})$. Similarily let Y be a vector field along γ represented by

$$Y(t) = \sum_{k=1}^{m} \sigma_k(t) \frac{\partial}{\partial x_k}|_{\gamma(t)}.$$

Then

$$\begin{aligned} \nabla_{\dot{\gamma}} Y(t) &= \sum_{j=1}^{m} \{\dot{\sigma}_j(t) \frac{\partial}{\partial x_j}|_{\gamma(t)} + \sigma_j(t) (\nabla_{\dot{\gamma}} \tfrac{\partial}{\partial x_j})_{\gamma(t)}\} \\ &= \sum_{k=1}^{m} \{\dot{\sigma}_k(t) + \sum_{i,j=1}^{m} \sigma_j(t) \rho_i(t) \Gamma_{ij}^k \circ \gamma(t)\} \frac{\partial}{\partial x_k}|_{\gamma(t)}. \end{aligned}$$

This implies that $\nabla_{\dot{\gamma}} Y \equiv 0$ if and only if

$$\dot{\sigma}_k(t) + \sum_{i,j=1}^{m} \sigma_j(t) \rho_i(t) \Gamma_{ij}^k \circ \gamma(t) = 0$$

for all $k = 1, \dots, m$. Since I is compact if follows from classical results on ODEs that to each initial value $\sigma(t_0) = (v_1, \dots, v_m) \in \mathbb{R}^m$ with

$$X_0 = \sum_{k=1}^{m} v_i \frac{\partial}{\partial x_k}|_{\gamma(t_0)}$$

there exists a unique solution $\sigma = (\sigma_1, \dots, \sigma_m)$ to the above system. This gives us the unique parallel vector field Y

$$Y(t) = \sum_{k=1}^m \sigma_k(t) \frac{\partial}{\partial x_k}|_{\gamma(t)}$$

along I. □

Theorem 8.8. *Let (M, g) be a Riemannian manifold. If $p \in M$ and $v \in T_pM$ then there exists an open interval $I = (-\epsilon, \epsilon)$ and a unique geodesic $\gamma : I \to M$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.*

PROOF. Let (U, x) be a local chart on M such that $p \in U$. For a C^2-curve $\gamma : J \to U$ we put $\gamma_i = x_i \circ \gamma : J \to \mathbb{R}$. The curve $x \circ \gamma : J \to \mathbb{R}^m$ is C^2 so we have

$$(dx)_{\gamma(t)}(\dot{\gamma}(t)) = \sum_{i=1}^m \dot{\gamma}_i(t) e_i.$$

This implies that

$$\dot{\gamma}(t) = \sum_{i=1}^m \dot{\gamma}_i(t) \frac{\partial}{\partial x_i}|_{\gamma(t)}.$$

By differentation we obtain

$$\begin{aligned} \nabla_{\dot{\gamma}}\dot{\gamma} &= \sum_{j=1}^m \nabla_{\dot{\gamma}} \dot{\gamma}_j(t) \tfrac{\partial}{\partial x_j}|_{\gamma(t)} \\ &= \sum_{j=1}^m \{\ddot{\gamma}_j(t) \frac{\partial}{\partial x_j}|_{\gamma(t)} + \sum_{i=1}^m \dot{\gamma}_j(t)\dot{\gamma}_i(t)(\nabla_{\frac{\partial}{\partial x_i}} \tfrac{\partial}{\partial x_j})_{\gamma(t)}\} \\ &= \sum_{k=1}^m \{\ddot{\gamma}_k(t) + \sum_{i,j=1}^m \dot{\gamma}_j(t)\dot{\gamma}_i(t)\Gamma_{ij}^k \circ \gamma(t)\} \frac{\partial}{\partial x_k}|_{\gamma(t)}. \end{aligned}$$

Hence the curve γ is a geodesic if and only if

$$\ddot{\gamma}_k(t) + \sum_{i,j=1}^m \dot{\gamma}_j(t)\dot{\gamma}_i(t)\Gamma_{ij}^k \circ \gamma(t) = 0$$

for all $k = 1, \dots, m$. It follows from classical results on ODEs that for initial values $q_0 = x(p)$ and $w_0 = (dx)_p(v)$ there exists an open interval $(-\epsilon, \epsilon)$ and an unique solution $(\gamma_1, \dots, \gamma_m)$ satisfying the initial conditions

$$(\gamma_1(0), \dots, \gamma_m(0)) = q_0 \quad \text{and} \quad (\dot{\gamma}_1(0), \dots, \dot{\gamma}_m(0)) = w_0.$$

□

Definition 8.9. A Riemannian manifold (M, g) is called **complete** if for each point $(p, v) \in TM$ there exists a geodesic $\gamma : \mathbb{R} \to M$ defined on the whole of $\mathbb{R}$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$.

Example 8.10. Let $(M, g) = E^m$ be the Euclidean space. For the trivial chart $\mathrm{id}_{\mathbb{R}^m} : \mathbb{R}^m \to \mathbb{R}^m$ the metric is given by $g_{ij} = \delta_{ij}$, so $\Gamma_{ij}^k = 0$ for all $i, j, k = 1, \dots, m$. This means that $\gamma : I \to \mathbb{R}^m$ is a geodesic if and only if $\ddot\gamma(t) = 0$ or equivalently $\gamma(t) = t \cdot a + b$ for some $a, b \in \mathbb{R}^m$. This proves that the geodesics are the straight lines.

Proposition 8.11. *Let (M, g) be a Riemannian manifold and $\tilde M$ be a submanifold equipped with the induced metric $\tilde g$. A curve $\gamma : I \to \tilde M$ is a geodesic in $\tilde M$ if and only if $(\nabla_{\dot\gamma}\dot\gamma)^T = 0$.*

PROOF. The statement follows directly from the fact that $\tilde\nabla_{\dot\gamma}\dot\gamma = (\nabla_{\dot\gamma}\dot\gamma)^T$. □

Example 8.12. Let E^{m+1} be the $(m+1)$-dimensional Euclidean space and $S^m \subset E^{m+1}$ be the unit sphere with the induced metric. At a point $p \in S^m$ the normal space N_pS^m is simply the line spanned by p. If $\gamma : I \to S^m$ is a curve on the sphere, then $\nabla_{\dot\gamma}\dot\gamma = \ddot\gamma^T = \ddot\gamma - \ddot\gamma^N = \ddot\gamma - \langle\ddot\gamma, \gamma\rangle\gamma$. This shows that γ is a geodesic if and only if

$$\ddot\gamma = \langle\ddot\gamma, \gamma\rangle\gamma. \tag{4}$$

For a point $(p, v) \in TS^m$ define the curve $\gamma = \gamma_{(p,v)} : \mathbb{R} \to S^m$ by

$$\gamma : t \mapsto \begin{cases} p & \text{if } v = 0 \\ \cos(|v|t) \cdot p + \sin(|v|t) \cdot v/|v| & \text{if } v \neq 0. \end{cases}$$

Then one easily checks that $\gamma(0) = p$, $\dot\gamma(0) = v$ and that γ satisfies the geodesic equation (4). This implies that

i. every geodesic on S^m is a great circle,
ii. the standard sphere is complete.

Example 8.13. Let $\mathrm{Sym}(\mathbb{R}^{m+1})$ be equipped with the metric

$$\langle A, B\rangle = \frac{1}{8}\mathrm{trace}(A^t \cdot B).$$

Then we know that the map $\phi : S^m \to \mathrm{Sym}(\mathbb{R}^{m+1})$ with

$$\phi : p \mapsto (2pp^t - I)$$

is an isometric immersion such that $\phi(S^m) \cong \mathbb{R}P^m$ i.e. the m-dimensional real projective space. The map ϕ is locally an isometry, so the geodesics on $\mathbb{R}P^m$ are exactly the images of geodesics on S^m. This shows that the real projective space is complete.

Definition 8.14. Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ be a C^r-curve on M. A **variation** of γ is a C^r-map $\Phi : (-\epsilon, \epsilon) \times I \to M$ such that for all $s \in I$, $\Phi_0(s) = \Phi(0, s) = \gamma(s)$. If the interval is compact i.e. of the form $I = [a, b]$, then the variation Φ is called **proper** if for all $t \in (-\epsilon, \epsilon)$, $\Phi_t(a) = \gamma(a)$ and $\Phi_t(b) = \gamma(b)$.

Definition 8.15. Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ be a C^2-curve on M. For every compact interval $[a, b] \subset I$ we define the energy functional $E_{[a,b]}$ by

$$E_{[a,b]}(\gamma) = \frac{1}{2}\int_a^b g(\dot{\gamma}(t), \dot{\gamma}(t))dt.$$

A C^2-curve $\gamma : I \to M$ is called **a critical point** for the energy functional if every proper variation Φ of $\gamma|_{[a,b]}$ satisfies

$$\frac{d}{dt}(E_{[a,b]}(\Phi_t))|_{t=0} = 0.$$

Theorem 8.16. *A C^2-curve γ is a critical point for the energy functional if and only if it is a geodesic.*

Proof. For a C^2-map $\Phi : (-\epsilon, \epsilon) \times I \to M$, $\Phi : (t, s) \mapsto \Phi(t, s)$ we define the vector fields $X = d\Phi(\partial/\partial s)$ and $Y = d\Phi(\partial/\partial t)$ along Φ. The following shows that the vector fields X and Y commute:

$$\nabla_X Y - \nabla_Y X = [X, Y] = [d\Phi(\partial/\partial s), d\Phi(\partial/\partial t)] = d\Phi([\partial/\partial s, \partial/\partial t]) = 0,$$

since $[\partial/\partial s, \partial/\partial t] = 0$.

We now assume that Φ is a proper variation of $\gamma|_{[a,b]}$. Then

$$\begin{aligned}
\frac{d}{dt}(E_{[a,b]}(\Phi_t)) &= \frac{1}{2}\frac{d}{dt}(\int_a^b g(X, X)ds) \\
&= \frac{1}{2}\int_a^b \frac{d}{dt}(g(X, X))ds \\
&= \int_a^b g(\nabla_Y X, X)ds \\
&= \int_a^b g(\nabla_X Y, X)ds \\
&= \int_a^b (\frac{d}{ds}(g(Y, X)) - g(Y, \nabla_X X))ds \\
&= [g(Y, X)]_a^b - \int_a^b g(Y, \nabla_X X)ds.
\end{aligned}$$

The variation is proper, so $Y(a) = Y(b) = 0$. Furthermore $X(0,s) = \partial\Phi/\partial s(0,s) = \dot\gamma(s)$, so

$$\frac{d}{dt}(E_{[a,b]}(\Phi_t))|_{t=0} = -\int_a^b g(Y(0,s),(\nabla_{\dot\gamma}\dot\gamma)(s))ds.$$

The last integral vanishes for every proper variation Φ of γ if and only if $\nabla_{\dot\gamma}\dot\gamma = 0$. □

Remark 8.17. A geodesic $\gamma : I \to (M,g)$ is a special case of what is called a harmonic map $\phi : (M,g) \to (N,h)$ between Riemannian manifolds. Other examples are conformal immersions $\psi : (M^2,g) \to (N,h)$ which parametrize the so called minimal surfaces in (N,h). For references on harmonic maps see

i. J. Eells, L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc. 10, (1978), 1-68.
ii. J. Eells, L. Lemaire, *Selected topics in harmonic maps*, CBMS Regional Conf. Ser. in Math. 50, AMS (1983).
iii. J. Eells, L. Lemaire, *Another report on harmonic maps*, Bull. London Math. Soc. 20, (1988), 385-524.
iv. J. Jost, *Harmonic maps - analytic theory and geometric significance*, in Lecture Notes in Math. 1357, Springer (1988)
iv. F.E. Burstall, L. Lemaire, J. Rawnsley, *The Harmonic Maps Bibliography*,
`http://www.bath.ac.uk/Departments/Maths/home.html`
v. S. Gudmundsson, *The Harmonic Morphisms Bibliography*,
`http://www.maths.lth.se/matematiklu/personal/sigma/harmonic/bibliography.html`

Let (M^m,g) be an m-dimensional Riemannian manifold, $p \in M$ and

$$S_p^{m-1} = \{v \in T_pM |\ g_p(v,v) = 1\}$$

be the unit sphere in the tangent space T_pM at p. Then every point $w \in T_pM - \{0\}$ can be written as $w = r_w \cdot v_w$, where $r_w = |w|$ and $v_w = w/|w| \in S_p^{m-1}$. For $v \in S_p^{m-1}$ let $\gamma_v : (-\alpha_v, \beta_v) \to M$ be the maximal geodesic such that $\alpha_v, \beta_v \in \mathbb{R}^+ \cup \{\infty\}$, $\gamma_v(0) = p$ and $\dot\gamma_v(0) = v$. Define

$$\epsilon_p = \inf\{\alpha_v, \beta_v |\ v \in S_p^{m-1}\}.$$

The unit sphere S_p^{m-1} is compact, so $\epsilon_p > 0$. Put

$$B_{\epsilon_p}^m(0) = \{v \in T_pM |\ g_p(v,v) < \epsilon_p^2\}.$$

Definition 8.18. For the above situation we define the **exponential map** $\exp_p : B^m_{\epsilon_p}(0) \to M$ at p by

$$\exp_p : w \mapsto \begin{cases} p & \text{if } w = 0 \\ \gamma_{v_w}(r_w) & \text{if } w \neq 0. \end{cases}$$

Note that for $v \in S^{m-1}_p$ the line segment $\lambda_v : (-\epsilon_p, \epsilon_p) \to T_pM$ with $\lambda_v : t \mapsto t \cdot v$ is mapped onto the geodesic γ_v i.e. locally we have $\gamma_v = \exp_p \circ \lambda_v$. One can prove that the map $\exp_p$ is smooth and it follows from its definition that the differential $d(\exp_p)_p : T_pM \to T_pM$ is the identity map for the tangent space T_pM. It then follows from the inverse mapping theorem that there exists an $r_p \in \mathbb{R}^+$ such that if $U_p = B^m_{r_p}(0)$ and $V_p = \exp_p(U_p)$ then $\exp_p|_{U_p} : U_p \to V_p$ is a diffeomorphism parametrizing the open subset V_p of M.

Theorem 8.19. *Let (M, g) be a Riemannian manifold. Then the geodesics are locally the shortest paths between their endpoints, or more precisely: If $p \in M$ and $\gamma : [0, \epsilon] \to M$ is a geodesic with $\gamma(0) = p$, then there exists an α with $0 < \alpha \leq \epsilon$ such that for each $q \in \gamma([0, \alpha])$, γ is the shortest path from p to q.*

Proof. Let $p \in M$, $U = B^m_r(0) \subset T_pM$ and $V = \exp_p(U)$ be such that the restriction $\phi = \exp_p|_U : U \to V$ of the exponential map at p is a diffeomorphism. On V we have the metric g which we pull back via ϕ to obtain $\tilde{g} = \phi^* g$ on U. This makes $\phi : (U, \tilde{g}) \to (V, g)$ into an isometry. It then follows from the construction of the exponential map, that the geodesics in $(U, \tilde{g})$ through the point $0 = \phi^{-1}(p)$ are exactly the lines $\lambda_v : t \to t \cdot v$ where $v \in T_pM$. Now let $q \in B^m_r(0) - \{0\}$ and $\lambda_q : [0, 1] \to B^m_r(0)$ be the curve $\lambda_q : t \mapsto t \cdot q$. Further let $\sigma : [0, 1] \to B^m_r(0)$ be any other curve such that $\sigma(0) = 0$ and $\sigma(1) = q$. Along σ we define two vector fields $\hat{\sigma}$ and $\dot{\sigma}_{\text{rad}}$ by $\hat{\sigma} : t \mapsto (\sigma(t), \sigma(t))$ and

$$\dot{\sigma}_{\text{rad}} : t \mapsto (\sigma(t), \frac{\tilde{g}_{\sigma(t)}(\dot{\sigma}(t), \hat{\sigma}(t))}{\tilde{g}_{\sigma(t)}(\hat{\sigma}(t), \hat{\sigma}(t))} \cdot \hat{\sigma}(t)).$$

Then it is easily checked that

$$|\dot{\sigma}_{\text{rad}}(t)| = \frac{|\tilde{g}_{\sigma(t)}(\dot{\sigma}(t), \hat{\sigma}(t))|}{|\hat{\sigma}|},$$

and

$$\frac{d}{dt}|\hat{\sigma}(t)| = \frac{d}{dt}\sqrt{\tilde{g}_{\sigma(t)}(\hat{\sigma}(t), \hat{\sigma}(t))} = \frac{\tilde{g}_{\sigma}(\dot{\sigma}, \hat{\sigma})}{|\hat{\sigma}|}.$$

Combining these two equations we obtain

$$|\dot{\sigma}_{\text{rad}}(t)| \geq \frac{d}{dt}|\hat{\sigma}(t)|.$$

This implies that

$$\begin{aligned} L(\sigma) &= \int_0^1 |\dot{\sigma}| dt \\ &\geq \int_0^1 |\dot{\sigma}_{\text{rad}}| dt \\ &\geq \int_0^1 \frac{d}{dt} |\hat{\sigma}(t)| dt \\ &= |\hat{\sigma}(1)| - |\hat{\sigma}(0)| \\ &= |q| \\ &= L(\lambda_q). \end{aligned}$$

This proves that in fact λ_q is the shortest path connecting p and q. □

Definition 8.20. Let (M, g) be a Riemannian manifold and $\tilde{M}$ be a submanifold with the induced metric $\tilde{g}$. Then

$$\mathrm{H} = \frac{1}{\tilde{m}} \text{trace}(B) \in C^\infty(N\tilde{M})$$

is called the **mean curvature vector field** of $\tilde{M}$ in M. The submanifold $\tilde{M}$ is said to be

i. **minimal** in M if $\mathrm{H} \equiv 0$, and
ii. **totally geodesic** if $B \equiv 0$.

Proposition 8.21. *Let (M, g) be a Riemannian manifold and $\tilde{M}$ be a submanifold equipped with the induced metric $\tilde{g}$. Then the following conditions are equivalent:*

i. *$\tilde{M}$ is totally geodesic in M*
ii. *if $\gamma : I \to \tilde{M}$ is a curve, then the following conditions are equivalent*
    a. *$\gamma : I \to \tilde{M}$ is a geodesic in $\tilde{M}$,*
    b. *$\gamma : I \to M$ is a geodesic in M.*

Proof. The result immediately follows from the following decomposition formula

$$\nabla_{\dot{\gamma}} \dot{\gamma} = (\nabla_{\dot{\gamma}} \dot{\gamma})^T + (\nabla_{\dot{\gamma}} \dot{\gamma})^N = \tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} + B(\dot{\gamma}, \dot{\gamma}).$$

□

Proposition 8.22. *Let (M, g) be a Riemannian manifold and $\tilde{M}$ be a submanifold. For an arbitrary $(p, v) \in T\tilde{M}$ let $\gamma_{(p,v)} : I \to M$ be the geodesic in M with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Then $\tilde{M}$ is totally geodesic in (M, g) if and only if $\gamma_{(p,v)}(I) \subset \tilde{M}$ for all $(p, v) \in T\tilde{M}$.*

PROOF. See Exercise 8.3. □

Proposition 8.23. *Let (M, g) be a Riemannian manifold and $\tilde{M}$ be a submanifold which is the fixpoint set of an isometry $\phi : M \to M$. Then $\tilde{M}$ is totally geodesic in M.*

PROOF. Let $p \in \tilde{M}$, $v \in T_p\tilde{M}$ and $\gamma : I \to M$ be the geodesic with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. The map $\phi : M \to M$ is an isometry so $\phi \circ \gamma : I \to M$ is a geodesic. The uniqueness result of Theorem 8.8, $\phi(\gamma(0)) = \gamma(0)$ and $d\phi(\dot{\gamma}(0)) = \dot{\gamma}(0)$ then imply that $\phi(\gamma) = \gamma$. Hence the image of the geodesic $\gamma : I \to M$ is contained in $\tilde{M}$, so following Proposition 8.22 $\tilde{M}$ is totally geodesic in M. □

Corollary 8.24. *If $\tilde{m} < m$ then the $\tilde{m}$-dimensional sphere*

$$S^{\tilde{m}} = \{(x, 0) \in \mathbb{R}^{\tilde{m}+1} \times \mathbb{R}^{m-\tilde{m}} |\ |x|^2 = 1\}$$

is totally geodesic in

$$S^m = \{(x, y) \in \mathbb{R}^{\tilde{m}+1} \times \mathbb{R}^{m-\tilde{m}} |\ |x|^2 + |y|^2 = 1\}.$$

PROOF. The submanifold $S^{\tilde{m}}$ of S^m is the fixpoint set of the isometry $\phi : S^m \to S^m$ with $(x, y) \mapsto (x, -y)$. □

Exercises

Exercise 8.1. Let $H^2 = (\mathbb{R}\times\mathbb{R}^+, \frac{1}{y^2}\langle,\rangle_{\mathbb{R}^2})$ be the hyperbolic plane. Find all geodesics in H^2.

Exercise 8.2. Let the orthogonal group $\mathbf{O}(n)$ be equipped with the left-invariant metric $g(A,B) = \text{trace}(A^t B)$. Prove that a C^2-curve $\gamma : (-\epsilon, \epsilon) \to \mathbf{O}(n)$ is a geodesic if and only if $\gamma^t \cdot \ddot{\gamma} = \ddot{\gamma}^t \cdot \gamma$.

Exercise 8.3. Find a proof for Proposition 8.22.

Exercise 8.4. Determine for which $\theta \in (0, \pi/2)$ the topological 2-torus

$$T^2_\theta = \{(\cos\theta e^{i\alpha}, \sin\theta e^{i\beta}) \in S^3 |\ \alpha, \beta \in \mathbb{R}\}$$

is a minimal submanifold of the 3-dimensional sphere

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 |\ |z_1|^2 + |z_2|^2 = 1\}.$$

Exercise 8.5. Show that $H^{\tilde{m}} = \{(x, 0) \in \mathbb{R}^{\tilde{m}} \times \mathbb{R}^{m-\tilde{m}} |\ |x| < 1\}$ is a totally geodesic submanifold of H^m.

Exercise 8.6. Determine the totally geodesic submanifolds of the m-dimensional real projective space $\mathbb{R}P^m$.

Exercise 8.7. Let the orthogonal group $\mathbf{O}(n)$ be equipped with the left-invariant metric $g(A,B) = \text{trace}(A^t B)$ and let $K \subset \mathbf{O}(n)$ be a Lie subgroup. Prove that K is totally geodesic in $\mathbf{O}(n)$.

CHAPTER 9

The Curvature Tensor

Definition 9.1. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. For tensor fields $A : C_r^\infty(TM) \to C_0^\infty(TM)$ and $B : C_r^\infty(TM) \to C_1^\infty(TM)$ we define their **covariant derivatives** $\nabla A : C_{r+1}^\infty(TM) \to C_0^\infty(TM)$ and $\nabla B : C_{r+1}^\infty(TM) \to C_1^\infty(TM)$ by

$$\nabla A : (X, X_1, \dots, X_r) \mapsto (\nabla_X A)(X_1, \dots, X_r) =$$

$$X(A(X_1, \dots, X_r)) - \sum_{i=1}^{r} A(X_1, \dots, X_{i-1}, \nabla_X X_i, X_{i+1}, \dots, X_r)$$

$$\nabla B : (X, X_1, \dots, X_r) \mapsto (\nabla_X B)(X_1, \dots, X_r) =$$

$$\nabla_X(B(X_1, \dots, X_r)) - \sum_{i=1}^{r} B(X_1, \dots, X_{i-1}, \nabla_X X_i, X_{i+1}, \dots, X_r).$$

A tensor field E of type $(r, 0)$ or $(r, 1)$ is said to be **parallel** if $\nabla E \equiv 0$. An example of a parallel tensor field of type $(2, 0)$ is the Riemannian metric g of (M, g). For this see Exercise 9.1.

Let (M, g) be a Riemannian manifold. A vector field $Z \in C^\infty(TM)$ defines a smooth tensor field $\hat{Z} : C_1^\infty(TM) \to C_1^\infty(TM)$ given by

$$\hat{Z} : X \mapsto \nabla_X Z.$$

For two vector fields $X, Y \in C^\infty(TM)$ we define the **second covariant derivative** $\nabla^2_{X,Y} : C_1^\infty(TM) \to C_1^\infty(TM)$ by

$$\nabla^2_{X,Y} : Z \mapsto (\nabla_X \hat{Z})(Y).$$

It then follows from the definition above that

$$\nabla^2_{X,Y} Z = \nabla_X(\hat{Z}(Y)) - \hat{Z}(\nabla_X Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z.$$

Definition 9.2. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Let $R : C_3^\infty(TM) \to C_1^\infty(TM)$ be twice the skew-symmetric part of the second covariant derivative ∇^2 i.e.

$$R(X, Y)Z = \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Then R is a smooth tensor field of type $(3, 1)$ which we call the **curvature tensor** of the Riemannian manifold (M, g).

Proof. See Exercise 9.2. □

Note that the curvature tensor R only depends on the intrinsic object ∇ and hence it is intrinsic itself.

Proposition 9.3. *Let (M, g) be a smooth Riemannian manifold. For vector fields X, Y, Z, W on M we then have*

i. $R(X, Y)Z = -R(Y, X)Z$,
ii. $g(R(X, Y)Z, W) = -g(R(X, Y)W, Z)$,
iii. $g(R(X, Y)Z, W) + g(R(Z, X)Y, W) + g(R(Y, Z)X, W) = 0$,
iv. $g(R(X, Y)Z, W) = g(R(Z, W)X, Y)$,
v. $6 \cdot R(X, Y)Z = R(X, Y + Z)(Y + Z) - R(X, Y - Z)(Y - Z) + R(X + Z, Y)(X + Z) - R(X - Z, Y)(X - Z)$.

Proof. See Exercise 9.3. □

For a point $p \in M$ let $G_2(T_pM)$ denote the set of all 2-dimensional subspaces of T_pM i.e.

$$G_2(T_pM) = \{V \subset T_pM |\ V \text{ is a 2-dimensional subspace of } T_pM\}.$$

Lemma 9.4. *Let $X, Y, Z, W \in T_pM$ such that the two 2-dimensional subspaces $span_{\mathbb{R}}\{X, Y\}, span_{\mathbb{R}}\{Z, W\} \in G_2(T_pM)$ are equal. Then*

$$\frac{g(R(X, Y)Y, X)}{|X|^2|Y|^2 - g(X, Y)^2} = \frac{g(R(Z, W)W, Z)}{|Z|^2|W|^2 - g(Z, W)^2}.$$

Proof. See Exercise 9.4. □

Definition 9.5. For a point $p \in M$ the function $K_p : G_2(T_pM) \to \mathbb{R}$ with

$$K_p : \text{span}_{\mathbb{R}}\{X, Y\} \mapsto \frac{g(R(X, Y)Y, X)}{|X|^2|Y|^2 - g(X, Y)^2}$$

is called the **sectional curvature** at p. Furthermore we define the functions $\delta, \Delta : M \to \mathbb{R}$ by

$$\delta : p \mapsto \min_{V \in G_2(T_pM)} K_p(V) \quad \text{and} \quad \Delta : p \mapsto \max_{V \in G_2(T_pM)} K_p(V).$$

The Riemannian manifold (M, g) is said to be

i. of (**strictly**) **positive curvature** if $\inf_{p \in M} \delta(p) \geq 0$ (> 0),
ii. of (**strictly**) **negative curvature** if $\sup_{p \in M} \Delta(p) \leq 0$ (< 0),
iii. of **constant curvature** if $\delta = \Delta$ is constant,
iv. **flat** if $\delta \equiv \Delta \equiv 0$.

Proposition 9.6. *Let (M,g) be a Riemannian manifold and let (U,x) be a local coordinate on M. For $i,j,k,l = 1,\dots,m$ put $X_i = \partial/\partial x_i$ and $R_{ijkl} = g(R(X_i,X_j)X_k,X_l)$. Then*

$$R_{ijkl} = \sum_{s=1}^m g^{sl}\Big(\frac{\partial\Gamma_{jk}^s}{\partial x_i} - \frac{\partial\Gamma_{ik}^s}{\partial x_j} + \sum_{r=1}^m\{\Gamma_{jk}^r\cdot\Gamma_{ir}^s - \Gamma_{ik}^r\cdot\Gamma_{jr}^s\}\Big)$$

Proof. Using the fact that $[X_i,X_j] = 0$ we obtain

$$\begin{aligned}
&R(X_i,X_j)X_k = \nabla_{X_i}\nabla_{X_j}X_k - \nabla_{X_j}\nabla_{X_i}X_k\\
&= \nabla_{X_i}(\textstyle\sum_{s=1}^m \Gamma_{jk}^s\cdot X_s) - \nabla_{X_j}(\sum_{s=1}^m \Gamma_{ik}^s\cdot X_s)\\
&= \sum_{s=1}^m\Big(\frac{\partial\Gamma_{jk}^s}{\partial x_i}\cdot X_s + \sum_{r=1}^m \Gamma_{jk}^s\Gamma_{is}^r X_r - \frac{\partial\Gamma_{ik}^s}{\partial x_j}\cdot X_s - \sum_{r=1}^m\Gamma_{ik}^s\Gamma_{js}^r X_r\Big)\\
&= \sum_{s=1}^m\Big(\frac{\partial\Gamma_{jk}^s}{\partial x_i} - \frac{\partial\Gamma_{ik}^s}{\partial x_j} + \sum_{r=1}^m\{\Gamma_{jk}^r\Gamma_{ir}^s - \Gamma_{ik}^r\Gamma_{jr}^s\}\Big)X_s.
\end{aligned}$$

□

Example 9.7. Let (M,g) be the Euclidean space. Then the set $\{\partial/\partial x_1,\dots,\partial/\partial x_m\}$ is a global frame for $T\mathbb{R}^m$. We have $g_{ij} = \delta_{ij}$, so $\Gamma_{ij}^k \equiv 0$. This implies that $R \equiv 0$ so E^m is flat.

Example 9.8. The standard sphere S^m has constant sectional curvature $+1$ (see Exercises 9.7 and 9.8) and the hyperbolic space H^m has constant sectional curvature -1 (see Exercise 9.9).

Our next goal is Corollary 9.12 where we obtain a formula for the curvature tensor of the manifolds of constant sectional curvature κ. This turns out to be very useful in the study of Jacobi fields later on.

Lemma 9.9. *Let (M,g) be a Riemannian manifold and $(p,Y) \in TM$. Then the map $\tilde{Y} : T_pM \to T_pM$ with $\tilde{Y} : X \mapsto R(X,Y)Y$ is a symmetric endomorphism of the tangent space T_pM.*

Proof. For $Z \in T_pM$ we have

$$\begin{aligned}
g(\tilde{Y}(X),Z) &= g(R(X,Y)Y,Z) = g(R(Y,Z)X,Y)\\
&= g(R(Z,Y)Y,X) = g(X,\tilde{Y}(Z)).
\end{aligned}$$

□

Lemma 9.9 implies the existence of eigenvectors $X_1,\dots,X_m$ for the symmetric endomorphism $\tilde{Y}$ which form an orthonormal basis for the tangent space T_pM such that the corresponding eigenvalues satisfy

$$\lambda_1(p) \le \dots \le \lambda_m(p).$$

Definition 9.10. Let (M,g) be a Riemannian manifold. Then define the smooth tensor field $R_1 : C_3^\infty(TM) \to C_1^\infty(TM)$ of type $(3,1)$ by

$$R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y.$$

Proposition 9.11. *Let (M,g) be a smooth Riemannian manifold and X,Y,Z be vector fields on M. Then*

i. $|R(X,Y)Y - \frac{\delta+\Delta}{2} R_1(X,Y)Y| \leq \frac{1}{2}(\Delta-\delta)|X||Y|^2$

ii. $|R(X,Y)Z - \frac{\delta+\Delta}{2} R_1(X,Y)Z| \leq \frac{2}{3}(\Delta-\delta)|X||Y||Z|$

Proof. Without loss of generality we can assume that $|X| = |Y| = |Z| = 1$. If $X = X^\perp + X^T$ with $X^\perp \perp Y$ and X^T is a multiple of Y then $R(X,Y)Z = R(X^\perp,Y)Z$ and $|X^\perp| \leq |X|$ so we can also assume that $X \perp Y$. Then $R_1(X,Y)Y = \langle Y,Y\rangle X - \langle X,Y\rangle Y = X$.

The first statement follows from the fact that the symmetric endomorphism of T_pM with

$$X \mapsto \{R(X,Y)Y - \frac{\Delta+\delta}{2} \cdot X\}$$

has eigenvalues in the interval $[\frac{\delta-\Delta}{2}, \frac{\Delta-\delta}{2}]$.

It is easily checked that the operator R_1 satisfies the conditions of Proposition 9.3 and hence $D = R - \frac{\Delta+\delta}{2} \cdot R_1$ as well. This implies that

$$\begin{aligned} 6 \cdot D(X,Y)Z &= D(X,Y+Z)(Y+Z) - D(X,Y-Z)(Y-Z) \\ &+ D(X+Z,Y)(X+Z) - D(X-Z,Y)(X-Z). \end{aligned}$$

The second statement then follows from

$$\begin{aligned} 6|D(X,Y)Z| &\leq \frac{1}{2}(\Delta-\delta)\{|X|(|Y+Z|^2 + |Y-Z|^2) \\ &\qquad\qquad +|Y|(|X+Z|^2 + |X-Z|^2)\} \\ &= \frac{1}{2}(\Delta-\delta)\{2|X|(|Y|^2+|Z|^2) + 2|Y|(|X|^2+|Z|^2)\} \\ &= 4(\Delta-\delta). \end{aligned}$$

□

Corollary 9.12. *Let (M,g) be a Riemannian manifold of constant curvature κ. Then the curvature tensor R is given by*

$$R(X,Y)Z = \kappa(\langle Y,Z\rangle X - \langle X,Z\rangle Y).$$

Proof. This follows directly from Proposition 9.11 by using $\Delta = \delta = \kappa$. □

Proposition 9.13. *Let $(G,\langle,\rangle)$ be a Lie group equipped with a left-invariant metric such that for all $X \in \mathfrak{g}$ the endomorphism $ad(X)$:*

$\mathfrak{g} \to \mathfrak{g}$ is skew-symmetric with respect to $\langle , \rangle$. Then for any left-invariant vector fields $X, Y, Z \in \mathfrak{g}$ the curvature tensor R is given by

$$R(X,Y)Z = -\frac{1}{4}[[X,Y],Z].$$

PROOF. See Exercise 9.6. □

Theorem 9.14 (The Gauss Equation)**.** *Let (M,g) be a Riemannian manifold and $\tilde{M}$ be a submanifold equipped with the induced metric $\tilde{g}$. Let $X, Y, Z, W \in C^\infty(TM)$ be vector fields extending $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W} \in C^\infty(T\tilde{M})$. Then*

$$\begin{aligned}\langle \tilde{R}(\tilde{X},\tilde{Y})\tilde{Z},\tilde{W}\rangle &= \langle R(X,Y)Z,W\rangle + \langle B(\tilde{Y},\tilde{Z}),B(\tilde{X},\tilde{W})\rangle \\ &\quad -\langle B(\tilde{X},\tilde{Z}),B(\tilde{Y},\tilde{W})\rangle.\end{aligned}$$

PROOF. Using the definitions of the curvature tensors R, $\tilde{R}$, the Levi-Civita connection $\tilde{\nabla}$ and the second fundamental form of $\tilde{M}$ in M we obtain

$$\begin{aligned}&\langle \tilde{R}(\tilde{X},\tilde{Y})\tilde{Z},\tilde{W}\rangle \\ =\ &\langle \tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z},\tilde{W}\rangle \\ =\ &\langle (\nabla_X(\nabla_Y Z - B(Y,Z)))^T - (\nabla_Y(\nabla_X Z - B(X,Z)))^T, W\rangle \\ &-\langle (\nabla_{[X,Y]}Z - B([X,Y],Z))^T, W\rangle \\ =\ &\langle \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W\rangle \\ &-\langle \nabla_X(B(Y,Z)) - \nabla_Y(B(X,Z)), W\rangle \\ =\ &\langle R(X,Y)Z,W\rangle + \langle B(Y,Z),B(X,W)\rangle - \langle B(X,Z),B(Y,W)\rangle.\end{aligned}$$

□

As a direct consequence we get the following.

Corollary 9.15. *Let (M,g) be a Riemannian manifold and $\tilde{M}$ be a totally geodesic submanifold. Let $X, Y, Z, W \in C^\infty(TM)$ be vector fields extending $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W} \in C^\infty(T\tilde{M})$. Then*

$$\langle \tilde{R}(\tilde{X},\tilde{Y})\tilde{Z},\tilde{W}\rangle = \langle R(X,Y)Z,W\rangle.$$

Exercises

Exercise 9.1. Let (M, g) be a Riemannian manifold. Prove that the tensor field g of type $(2,0)$ is parallel with respect to the Levi-Civita connection.

Exercise 9.2. Let (M, g) be a Riemannian manifold. Prove that R is a smooth tensor field of type $(3,1)$.

Exercise 9.3. Find a proof for Proposition 9.3.

Exercise 9.4. Find a proof for Lemma 9.4.

Exercise 9.5. Let g be the Euclidean metric on $\mathbb{C}^m$ given by

$$g(z, w) = \sum_{k=1}^{m} \text{Re}(z_k \bar{w}_k).$$

Let T^m be the m-dimensional torus $\{z \in \mathbb{C}^m |\ |z_1| = ... = |z_m| = 1\}$ with the induced metric $\tilde{g}$.

i. Find an isometric immersion $\phi : \mathbb{R}^m \to T^m$.
ii. Determine all geodesics on (T^m, g).
iii. Prove that (T^m, g) is flat.

Exercise 9.6. Find a proof for Proposition 9.13.

Exercise 9.7. Let the Lie group $S^3 \cong \mathbf{SU}(2)$ be equipped with the metric $\langle X, Y \rangle = \frac{1}{2}\text{Re}\{\text{trace}(\bar{X}^t \cdot Y)\}$.

i. Find an orthonormal basis for $T_e\mathbf{SU}(2)$.
ii. Show that $(\mathbf{SU}(2), g)$ has constant sectional curvature $+1$.

Exercise 9.8. Let S^m be the unit sphere in $\mathbb{R}^{m+1}$ equipped with the standard Euclidean metric $\langle , \rangle_{\mathbb{R}^{m+1}}$. Use the results of Corollaries 8.24, 9.15 and Exercise 9.7 to prove that $(S^m, \langle , \rangle_{\mathbb{R}^{m+1}})$ has constant sectional curvature $+1$

Exercise 9.9. Let $H^m = (\mathbb{R}^+ \times \mathbb{R}^{m-1}, \frac{1}{x_1^2}\langle , \rangle_{\mathbb{R}^m})$ be the m-dimensional hyperbolic space. On H^m we define the operation $*$ by $(\alpha, x) * (\beta, y) = (\alpha \cdot \beta, \alpha \cdot y + x)$. For $k = 1, \dots, m$ define the vector field $X_k \in C^\infty(TH^m)$ by $(X_k)_x = x_1 \cdot \frac{\partial}{\partial x_k}$. Prove that,

i. $(H^m, *)$ is a Lie group,
ii. the vector fields $X_1, \dots, X_m$ are left-invariant,
iii. the metric g is left-invariant,
iv. (H^m, g) has constant curvature -1.

CHAPTER 10

Curvature and Local Geometry

This chapter is devoted to the study of the local geometry of a Riemannian manifold and how that is controlled by its curvature tensor. We are interested in understanding the spreading of geodesics that all go through the same given point. Using Jacobi fields we obtain a fundamental comparison result describing the curvature dependence of local distances.

Definition 10.1. Let (M, g) be a smooth Riemannian manifold. By a smooth **1-parameter family of geodesics** we mean a C^∞-map

$$\Phi : (-\epsilon, \epsilon) \times I \to M$$

such that the curve $\gamma_{t_0} : I \to M$ given by $\gamma_{t_0} : s \mapsto \Phi(t_0, s)$ is a geodesic for all $t_0 \in (-\epsilon, \epsilon)$. We call $t \in (-\epsilon, \epsilon)$ the **family parameter** of Φ.

Proposition 10.2. *Let (M, g) be a Riemannian manifold and $\Phi :$ $(-\epsilon, \epsilon) \times I \to M$ be a 1-parameter family of geodesics. Then the vector field $J_{t_0} : I \to C^\infty_{\gamma_{t_0}}(TM)$ along γ_{t_0} given by*

$$J_{t_0}(s) = \frac{\partial \Phi}{\partial t}(t_0, s)$$

satisfies the second order ordinary differential equation

$$\nabla_{\dot\gamma_{t_0}} \nabla_{\dot\gamma_{t_0}} J_{t_0} + R(J_{t_0}, \dot\gamma_{t_0})\dot\gamma_{t_0} = 0.$$

Proof. Along Φ we put $X(t, s) = \partial\Phi/\partial s$ and $J(t, s) = \partial\Phi/\partial t$. Then $[\partial/\partial t, \partial/\partial s] = 0$ leads to

$$[J, X] = [d\Phi(\partial/\partial t), d\Phi(\partial/\partial s)] = d\Phi([\partial/\partial t, \partial/\partial s]) = 0.$$

The definition of the curvature tensor now implies that

$$\begin{aligned} R(J, X)X &= \nabla_J \nabla_X X - \nabla_X \nabla_J X - \nabla_{[J,X]} X \\ &= -\nabla_X \nabla_J X \\ &= -\nabla_X \nabla_X J. \end{aligned}$$

Hence for each $t_0 \in (-\epsilon, \epsilon)$ we have

$$\nabla_{\dot\gamma_{t_0}} \nabla_{\dot\gamma_{t_0}} J_{t_0} + R(J_{t_0}, \dot\gamma_{t_0})\dot\gamma_{t_0} = 0.$$

□

Definition 10.3. Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ be a geodesic. A vector field J along γ is called a **Jacobi field** if

$$\nabla_X \nabla_X J + R(J, X)X = 0$$

along γ where $X = \dot{\gamma}$. We denote the space of all Jacobi fields along γ by $\mathcal{J}_\gamma(TM)$.

Lemma 10.4. *Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ be a geodesic. Then the space of Jacobi fields $\mathcal{J}_\gamma(TM)$ is a vector space.*

Proof. This follows directly from the fact that the Jacobi field equation is linear in J. □

Proposition 10.5. *Let $\gamma : I \to M$ be a geodesic, $t_0 \in I$, $p = \gamma(t_0)$ and $X = \dot{\gamma}$ along γ. If $v, w \in T_pM$ are two tangent vectors at p then there exists a unique Jacobi field J along γ, such that $J_p = v$ and $(\nabla_X J)_p = w$.*

Proof. Let $\{X_1, \dots, X_m\}$ be an orthonormal frame of parallel vector fields along γ. If J is a vector field along γ, then $J = \sum_{i=1}^m a_i X_i$ where $a_i = \langle J, X_i \rangle$ are smooth functions on I. The vector fields $X_1, \dots, X_m$ are parallel so we have $\nabla_X J = \sum_{i=1}^m \dot{a}_i X_i$ and $\nabla_X \nabla_X J = \sum_{i=1}^m \ddot{a}_i X_i$. For the curvature tensor we have $R(X_i, X)X = \sum_{k=1}^m b_i^k X_k$, where $b_i^k = \langle R(X_i, X)X, X_k \rangle$ are smooth functions on I depending on the geometry of (M, g). This means that $R(J, X)X$ is given by $R(J, X)X = \sum_{i,k=1}^m a_i b_i^k X_k$. We now see that J is a Jacobi field if and only if

$$\sum_{i=1}^m (\ddot{a}_i + \sum_{k=1}^m a_k b_k^i) X_i = 0.$$

This is equivalent to the second order system

$$\ddot{a}_i + \sum_{k=1}^m a_k b_k^i = 0 \quad \text{for all } i = 1, 2, \dots, m,$$

of linear ODEs in $a = (a_1, \dots, a_m)$. A global solution will always exist and is uniquely determined by $a(t_0)$ and $\dot{a}(t_0)$. This implies that J exists globally and is uniquely determined by the initial conditions $J_{\gamma(t_0)}$ and $(\nabla_X J)_{\gamma(t_0)}$. □

Corollary 10.6. *Let $\gamma : I \to M^m$ be a geodesic. Then the vector space $\mathcal{J}_\gamma(TM)$ of all Jacobi fields along γ is $2m$-dimensional.*

Lemma 10.7. *Let (M, g) be a Riemannian manifold, $\gamma : I \to M$ be a geodesic and J be a Jacobi field along γ. Let $\lambda \in \mathbb{R}^*$ and $\sigma : \lambda I \to I$ be given by $\sigma : t \mapsto t/\lambda$, then $\gamma \circ \sigma : \lambda I \to M$ is a geodesic and $J \circ \sigma$ is a Jacobi field along $\gamma \circ \sigma$.*

PROOF. See Exercise 10.1. □

This means that when proving results about Jacobi fields along a geodesic γ we can always without loss of generality assume that $|\dot{\gamma}| = 1$.

Proposition 10.8. *Let (M, g) be a Riemannian manifold, $\gamma : I \to M$ be a geodesic with $|\dot{\gamma}| = 1$ and J be a Jacobi field along γ. Let J^T be the tangential part of J given by $J^T = \langle J, \dot{\gamma}\rangle\dot{\gamma}$ and $J^N = J - J^T$ be the normal part. Then J^T and J^N are Jacobi fields along γ and there exist $a, b \in \mathbb{R}$ such that $J^T(s) = (as + b)\dot{\gamma}(s)$ for all $s \in I$.*

PROOF. We now have

$$\begin{aligned}
\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}J^T + R(J^T, \dot{\gamma})\dot{\gamma} &= \nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}(\langle J, \dot{\gamma}\rangle\dot{\gamma}) + R(\langle J, \dot{\gamma}\rangle\dot{\gamma}, \dot{\gamma})\dot{\gamma} \\
&= \langle\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}J, \dot{\gamma}\rangle\dot{\gamma} \\
&= -\langle R(J, \dot{\gamma})\dot{\gamma}, \dot{\gamma}\rangle\dot{\gamma} \\
&= 0.
\end{aligned}$$

This shows that the tangential part J^T of J is a Jacobi field. The fact that $\mathcal{J}_\gamma(TM)$ is a vector space implies that the normal part $J^N = J - J^T$ of J also is a Jacobi field.

By differentiating $\langle J, \dot{\gamma}\rangle$ twice along γ we obtain

$$\frac{d^2}{ds^2}\langle J, \dot{\gamma}\rangle = \langle\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}J, \dot{\gamma}\rangle = -\langle R(J, \dot{\gamma})\dot{\gamma}, \dot{\gamma}\rangle = 0$$

so $\langle J, \dot{\gamma}\rangle(s) = (as + b)$ for some $a, b \in \mathbb{R}$. □

Note that the last statement of Proposition (10.8) implies that we now know all the tangential Jacobi fields along γ.

At this point we remind the reader of the following classical fact:

Remark 10.9. If κ is a real number then the unique solution to the initial value problem

$$\ddot{f} + \kappa \cdot f = 0, \quad f(0) = a \text{ and } \dot{f}(0) = b$$

is the function $f : \mathbb{R} \to \mathbb{R}$, $f(s) = ac_\kappa(s) + bs_\kappa(s)$ where $c_\kappa : \mathbb{R} \to \mathbb{R}$ and $s_\kappa : \mathbb{R} \to \mathbb{R}$ are given by

$$c_\kappa(s) = \begin{cases} cosh(\sqrt{|\kappa|}s) & \text{if } \kappa < 0, \\ 1 & \text{if } \kappa = 0, \\ cos(\sqrt{\kappa}s) & \text{if } \kappa > 0. \end{cases}$$

and

$$s_\kappa(s) = \begin{cases} sinh(\sqrt{|\kappa|}s)/\sqrt{|\kappa|} & \text{if } \kappa < 0, \\ s & \text{if } \kappa = 0, \\ sin(\sqrt{\kappa}s)/\sqrt{\kappa} & \text{if } \kappa > 0. \end{cases}$$

Example 10.10. Let $\mathbb{C}$ be the complex plane with the standard Euclidean metric $\langle,\rangle_{\mathbb{R}^2}$ of constant sectional curvature $\kappa = 0$. The rotations about the origin produce a 1-parameter family of geodesics $\Phi_t : s \mapsto se^{it}$. Along the geodesic $\gamma_0 : s \mapsto s$ we get the Jacobi field $J_0(s) = \partial\Phi_t/\partial t(0,s) = is$ with $|J_0(s)| = |s| = |s_\kappa(s)|^2$.

Example 10.11. Let S^2 be the unit sphere in the standard Euclidean 3-space $\mathbb{C} \times \mathbb{R}$ with the induced metric of constant sectional curvature $\kappa = +1$. Rotations about the $\mathbb{R}$-axis produce a 1-parameter family of geodesics $\Phi_t : s \mapsto (sin(s)e^{it}, cos(s))$. Along the geodesic $\gamma_0 : s \mapsto (sin(s), cos(s))$ we get the Jacobi field $J_0(s) = \partial\Phi_t/\partial t(0,s) = (isin(s), 0)$ with $|J_0(s)|^2 = sin^2(s) = |s_\kappa(s)|^2$.

Example 10.12. Let $B_1^2(0)$ be the open unit disk in the complex plane with the hyperbolic metric $4/(1-|z|^2)^2\langle,\rangle_{\mathbb{R}^2}$ of constant sectional curvature $\kappa = -1$. Rotations about the origin produce a 1-parameter family of geodesics $\Phi_t : s \mapsto tanh(s)e^{it}$. Along the geodesic $\gamma_0 : s \mapsto tanh(s)$ we get the Jacobi field $J_0(s) = i \cdot tanh(s)$ with

$$|J_0(s)|^2 = \frac{4 \cdot tanh^2(s)}{1 - tanh^2(s)} = sinh^2(s) = |s_\kappa(s)|^2.$$

We will now see that when the manifold (M, g) has constant sectional curvature we can completely solve the Jacobi field equation

$$\nabla_X \nabla_X J + R(J, X)X = 0$$

along any given geodesic $\gamma : I \to M$, where $X = \dot{\gamma}$.

Let (M, g) be a Riemannian manifold of constant sectional curvature κ and $\gamma : I \to M$ be a geodesic with $|X| = 1$ where $X = \dot{\gamma}$. Further let $P_1, P_2, \dots, P_{m-1}$ be parallel vector fields along γ such that $g(P_i, P_j) = \delta_{ij}$ and $g(P_i, X) = 0$. Any vector field J along γ may now be written as

$$J(s) = \sum_{i=1}^{m-1} f_i(s)P_i(s) + f_m(s)X(s).$$

We now see that J is a Jacobi field if and only if

$$\begin{aligned}
\sum_{i=1}^{m-1} \ddot{f}_i(s)P_i(s) + \ddot{f}_m(s)X(s) &= \nabla_X\nabla_X J \\
&= -R(J,X)X \\
&= -R(J^N,X)X \\
&= -\kappa(g(X,X)J^N - g(J^N,X)X) \\
&= -\kappa J^N \\
&= -\kappa\sum_{i=1}^{m-1} f_i(s)P_i(s).
\end{aligned}$$

This implies that J is a Jacobi field if and only if the following system of ODEs is satisfied:

$$\ddot{f}_i(s) + \kappa f_i(s) = 0 \ \text{ for all } i = 1,2,\dots,m-1 \text{ and } \ddot{f}_m(s) = 0. \tag{5}$$

It is clear that for the initial values

i. $J(s_0) = \sum_{i=1}^{m-1} v_i P_i(s_0) + v_m X(s_0)$,
ii. $(\nabla_X J)(s_0) = \sum_{i=1}^{m-1} w_i P_i(s_0) + w_m X(s_0)$

or equivalently

$$f_i(s_0) = v_i \ \text{ and } \ \dot{f}_i(s_0) = w_i \ \text{ for all } \ i = 1,2,\dots,m$$

we can solve the system (5) explicitly on the whole of I.

Corollary 10.13. *Let (M,g) be a Riemannian manifold, $\gamma : I \to M$ be a geodesic and J be a Jacobi field along γ. If $g(J(t_0),\dot\gamma(t_0)) = 0$ and $g((\nabla_{\dot\gamma}J)(t_0),\dot\gamma(t_0)) = 0$ for some $t_0 \in I$, then $g(J(t),\dot\gamma(t)) = 0$ for all $t \in I$.*

Proof. The conditions $v_m = w_m = 0$ imply that $f_m = 0$. □

Example 10.14. Let S^2 be the unit sphere in the standard Euclidean 3-space $\mathbb{C}\times\mathbb{R}$ with the induced metric of constant curvature $\kappa = +1$ and $\gamma : \mathbb{R} \to S^2$ be the geodesic given by $\gamma : s \mapsto (e^{is}, 0)$. Then $\dot\gamma(s) = (ie^{is}, 0)$ so it follows from Proposition (10.8) that all Jacobi fields tangential to γ are given by

$$J^T_{(a,b)}(s) = (as+b)(ie^{is},0) \quad \text{for some } a,b \in \mathbb{R}.$$

The vector field $P : \mathbb{R} \to TS^2$ given by $s \mapsto [(e^{is},0),(0,1)]$ satisfies $\langle P,\dot\gamma\rangle = 0$ and $|P| = 1$. The sphere S^2 is 2-dimensional and $\dot\gamma$ is parallel along γ so P must be parallel. This implies that all the Jacobi fields orthogonal to $\dot\gamma$ are given by

$$J^N_{(a,b)}(s) = (0, a\cos s + b\sin s) \ \text{ for some } a,b \in \mathbb{R}.$$

We will now see how Jacobi fields can be constructed in the general situation when the curvature not necessarily is constant: Let (M, g) be a complete Riemannian manifold, $p \in M$ and $v, w \in T_pM$. Then $s \mapsto s(v+tw)$ defines a 1-parameter family of lines in the tangent space T_pM which all pass through the origin $0 \in T_pM$. Remember that the exponential map

$$(\exp)_p|_{B^m_{\varepsilon_p(0)}} : B^m_{\varepsilon_p(0)} \to \exp(B^m_{\varepsilon_p(0)})$$

maps lines in T_pM through the origin onto geodesics on M. Hence the map

$$\Phi_t : s \mapsto (\exp)_p(s(v + tw))$$

is a 1-parameter family of geodesics through $p \in M$, as long as $s(v+tw)$ is an element of $B^m_{\varepsilon_p(0)}$. This means that $J : s \mapsto (\partial\Phi_t/\partial t)(0, s)$ is a Jacobi field along the geodesic $\gamma : s \mapsto \Phi_0(s)$ with $\gamma(0) = p$ and $\dot\gamma(0) = w$. It is easily verified that J satisfies the initial conditions $J(0) = 0$ and $(\nabla_X J)(0) = w$.

Lemma 10.15. *Let (M, g) be a Riemannian manifold with sectional curvature K uniformly bounded above by Δ and $\gamma : [0, \alpha] \to M$ be a geodesic on M with $|X| = 1$ and $X = \dot\gamma$. Further let $J : [0, \alpha] \to TM$ be a Jacobi field along γ such that $g(J, X) = 0$ and $|J| \neq 0$ on $(0, \alpha)$. Then*

i. $\frac{d^2}{ds^2}(|J|) + \Delta \cdot |J| \geq 0$,
ii. *if $f : [0, \alpha] \to \mathbb{R}$ is a C^2-function, such that*
   a. $\ddot f + \Delta \cdot f = 0$ *and* $f > 0$ *on* $(0, \alpha)$,
   b. $f(0) = |J(0)|$, *and*
   c. $\dot f(0) = |\nabla_X J(0)|$,

   then $f(s) \leq |J(s)|$ on $(0, \alpha)$,
iii. *if $J(0) = 0$, then $|\nabla_X J(0)| \cdot s_\Delta(s) \leq |J(s)|$ for all $s \in (0, \alpha)$.*

Proof. i. Using the facts that $|X| = 1$ and $\langle X, J\rangle = 0$ we obtain

$$\begin{aligned}
\frac{d^2}{ds^2}(|J|) &= \frac{d^2}{ds^2}\sqrt{\langle J, J\rangle} = \frac{d}{ds}\Big(\frac{\langle \nabla_X J, J\rangle}{|J|}\Big) \\
&= \frac{\langle \nabla_X \nabla_X J, J\rangle}{|J|} + \frac{|\nabla_X J|^2 |J|^2 - \langle \nabla_X J, J\rangle^2}{|J|^3} \\
&\geq \frac{\langle \nabla_X \nabla_X J, J\rangle}{|J|} \\
&= -\frac{\langle R(J, X)X, J\rangle}{|J|} \\
&\geq -\Delta \cdot |J|.
\end{aligned}$$

ii. Define the function $h : [0, \alpha) \to \mathbb{R}$ by $h : s \mapsto \frac{|J(s)|}{f(s)}$ for $s \in (0, \alpha)$ and $h(0) = \lim_{s \to 0} \frac{|J(s)|}{f(s)} = 1$. Then

$$\begin{aligned}
\dot{h}(s) &= \frac{1}{f^2(s)}\left(\frac{d}{ds}(|J(s)|)f(s) - |J(s)|\dot{f}(s)\right) \\
&= \frac{1}{f^2(s)}\int_0^s \left(\frac{d^2}{dt^2}(|J(t)|)f(t) - |J(t)|\ddot{f}(t)\right)dt \\
&= \frac{1}{f^2(s)}\int_0^s f(t)\left(\frac{d^2}{dt^2}(|J(t)|) + \Delta \cdot |J(t)|\right)dt \\
&\geq 0.
\end{aligned}$$

This implies that $\dot{h}(s) \geq 0$ so $f(s) \leq |J(s)|$ for all $s \in (0, \alpha)$.

iii. The function $f(s) = |(\nabla_X J)(0)| \cdot s_\Delta(s)$ satisfies $\ddot{f}(s) + \Delta f(s) = 0$, $f(0) = |J(0)| = 0$ and $\dot{f}(0) = |(\nabla_X J)(0)|$ so it follows from (ii) that $|(\nabla_X J)(0)| \cdot s_\Delta(s) = f(s) \leq |J(s)|$. □

Let (M, g) be a Riemannian manifold of sectional curvature which is uniformly bounded above, i.e. there exists a $\Delta \in \mathbb{R}$ such that $K_p(V) \leq \Delta$ for all $V \in G_2(T_pM)$ and $p \in M$. Let (M_Δ, g_Δ) be another Riemannian manifold which is complete and of constant sectional curvature $K \equiv \Delta$. Let $p \in M$, $p_\Delta \in M_\Delta$ and identify $T_pM \cong \mathbb{R}^m \cong T_{p_\Delta}M_\Delta$.

Let U be an open neighbourhood of $\mathbb{R}^m$ around 0 such that the exponential maps $(\exp)_p$ and $(\exp)_{p_\Delta}$ are diffeomorphisms from U onto their images $(\exp)_p(\mathrm{U})$ and $(\exp)_{p_\Delta}(U)$, respectively. Let (r, p, q) be a geodesic triangle i.e. a triangle with sides which are shortest paths between their end points. Furthermore let $c : [a, b] \to M$ be the side connecting r and q and $v : [a, b] \to T_pM$ be the curve defined by $c(t) = (\exp)_p(v(t))$. Put $c_\Delta(t) = (\exp)_{p_\Delta}(v(t))$ for $t \in [a, b]$ and then it directly follows that $c(a) = r$ and $c(b) = q$. Finally put $r_\Delta = c_\Delta(a)$ and $q_\Delta = c_\Delta(b)$.

Theorem 10.16. *For the above situation the following inequality for the distance function is satisfied:* $d(q_\Delta, r_\Delta) \leq d(q, r)$.

Proof. Define a 1-parameter family $s \mapsto s \cdot v(t)$ of straight lines in T_pM through p. Then $\Phi_t : s \mapsto (\exp)_p(s \cdot v(t))$ and $\Phi_t^\Delta : s \mapsto (\exp)_{p_\Delta}(s \cdot v(t))$ are 1-parameter families of geodesics through $p \in M$, and $p_\Delta \in M_\Delta$, respectively. Hence $J_t = \partial\Phi_t/\partial t$ and $J_t^\Delta = \partial\Phi_t^\Delta/\partial t$ are Jacobi fields satisfying the initial conditions $J_t(0) = J_t^\Delta(0) = 0$ and

$(\nabla_X J_t)(0) = (\nabla_X J_t^\Delta)(0) = \dot{v}(t)$. Using Lemma (10.15) we now obtain

$$\begin{aligned} |\dot{c}_\Delta(t)| &= |J_t^\Delta(1)| = |(\nabla_X J_t^\Delta)(0)| \cdot s_\Delta(1) \\ &= |(\nabla_X J_t)(0)| \cdot s_\Delta(1) \leq |J_t(1)| = |\dot{c}(t)| \end{aligned}$$

The curve c is the shortest path between r and q so we have

$$d(r_\Delta, q_\Delta) \leq L(c_\Delta) \leq L(c) = d(r, q).$$

□

We now add the assumption that the sectional curvature of the manifold (M, g) is uniformly bounded below i.e. there exists a $\delta \in \mathbb{R}$ such that $\delta \leq K_p(V)$ for all $V \in G_2(T_pM)$ and $p \in M$. Let (M_δ, g_δ) be a complete Riemannian manifold of constant sectional curvature δ. Let $p \in M$ and $p_\delta \in M_\delta$ and identify $T_pM \cong \mathbb{R}^m \cong T_{p_\delta}M_\delta$. Then a similar construction as above gives two pairs of points $q, r \in M$ and $q_\delta, r_\delta \in M_\delta$ and shows that

$$d(q, r) \leq d(q_\delta, r_\delta).$$

Combining these two results we obtain **locally**

$$d(q_\Delta, r_\Delta) \leq d(q, r) \leq d(q_\delta, r_\delta).$$

Exercises

Exercise 10.1. Find a proof for Lemma (10.7).

Exercise 10.2. Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ be a geodesic such that $X = \dot{\gamma} \neq 0$. Further let J be a non-vanishing Jacobi field along γ with $g(X, J) = 0$. Prove that if $g(J, J)$ is constant along γ then (M, g) does not have strictly negative curvature.